MATHEMATICAL

QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES,"

AND

TWO APPENDICES.

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W. J. C. MILLER, B.A.,

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Solved Questions.

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APPENDIX II.

Solutions of Questions mainly relating to the Theory of Probability and Averages, by Professor Zerr, M.A.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

10554. (Professor SYLVESTER.) — There are $\frac{1}{2}n(n-1)$ unknown quantities expressed by the binary combinations of n *umbrae* $1, 2, 3, \dots n$. Let us understand that $r.r = 0$, and write down the n equations

$$r.1+r.2+r.3+\dots+r.n = C_r \quad (r = 1, 2, 3, \dots n).$$

Prove that, when all the C's are positive, the sufficient and necessary condition in order that these equations may be soluble in positive quantities is that no one of the n quantities C shall be greater than the sum of all the others; and, furthermore, if this condition is satisfied, and all the C's are positive integers, one or more solutions of the equation system can always be found in positive integers.

Solution by H. J. WOODALL.

Since all the quantities are to be positive, we must have $2C_k \not> \Sigma_1^n C_k$, if $2C_k = \Sigma_1^n C_k$; then $p.q=0$, if neither p nor $q=k$. The question, then, is whether the condition is sufficient. n quantities are known to determine $\frac{1}{2}n(n-1)$ *umbrae*, and no two of the C's contain more than one each similar quantities; *i.e.*, Cp and Cq contain $p.q$ and $q.p$ respectively, which are equal by hypothesis (combinations).

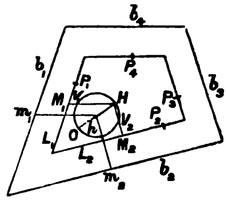
Hence there is sufficient latitude for the selection of suitable values of the *umbrae*. The remainder of the question is also an identity.

It seems to be necessary, if the *umbrae* are to be integers, that the sum of $C_1+C_2+\dots+C_n$ must be an even integer.

10723. (Professor SCHOUTE.)—In a plane four directly similar figures are given under the condition that any set of homologous lines l_1, l_2, l_3, l_4, taken in a determinate order, form a quadrilateral inscribed in a circle.

To show that the centre of this circle describes a limaçon of PASCAL, when l_1, l_2, l_3, l_4 envelope homologous circles of the similar figures.

Solution by Professor RAMASWAMI AIYAR.



Let P_1, P_2, P_3, P_4 be the centres of the circles; through them draw lines L_1, L_2, &c. parallel to b_1, b_2, &c., in any position; let h, H be the centres of the circles about the quadrilaterals. We shall first show that the locus of H is a circle.

Let M_1, M_2, &c. be the middle points of L_1, L_2, &c. Then, since L_1, L_2, L_4 pass through P_1, P_2, P_4, and form a triangle given in species, it is easily proved that the middle point M_1 of L_1 describes a circle passing through P_1; therefore M_1H, which is perpendicular to L_1, passes through the fixed point V_1 which is diametrically opposite to P_1 in this circle.

Similarly, M_2H passes through a fixed point V_2; and the angle V_1HV_2 being given, the locus of H is the circle about V_1HV_2.

Next, let m_1, m_2, &c. be the middle points of b_1, b_2, &c. Now it is easy to see that m_1h, m_2h are at constant distances from M_1H, M_2H; therefore Hh is a constant length, and Hh produced meets the circle V_1HV_2 in a fixed point O.

Therefore the locus of h is a limaçon of PASCAL of which O is the pole.

4258. (Professor EVANS, M.A.)—A person draws 10 cards from a full pack; find the chance that he has 30 spots.

Solution by Professor PUTNAM, M.A.

This problem is complicated by reason of the four suits. I have, as yet, been unable to bring it within the scope of the usual formula for "permutations and combinations," nor to discover any short method of solution, though I think one can be developed. By a laborious process, which I would not have undertaken had I foreseen the work required, I find the chance to be, in round numbers, one out of fifty-two.

Thirty spots can be drawn by drawing three tens. As there are four tens, these can be drawn in four ways. With these three cards there will be seven face-cards out of a total of twelve. The number of combinations of 12 cards taken 7 at a time is 792. Thirty spots can therefore be drawn with three 10-spots in $4 \times 792 = 3{,}168$ ways.

The next lower combination will be two 10's, one 9 and one 1. Two 10's can be taken in 6 ways, one 9 in 4 ways, and one 1 in 4 ways. Two 10's, one 9, and one 1 can be taken in $6 \times 4 \times 4 = 96$ ways. Here four spot-cards are used, which are to be combined with six face-cards. Twelve face cards, taken six at a time, give 924 combinations. Thirty spots can therefore be drawn with two 10's, one 9, and one 1, in $96 \times 924 = 88{,}704$ ways.

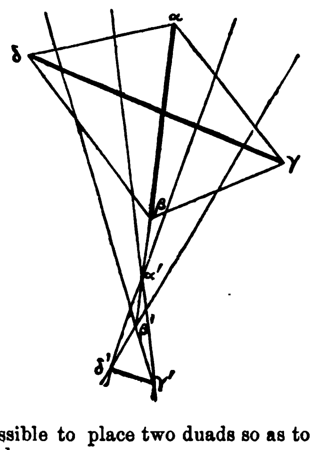
This process continued gives the total number of ways in which

30 spots can be drawn by drawing ten cards, 303,853,272. The total number of combinations of 52 cards taken 10 at a time is

$$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} = 15{,}820{,}024{,}220.$$

10716. (Professor CAYLEY, F.R.S.)—In a hexahedron ABCDA′B′C′D′ the plane faces of which are ABCD, A′B′C′D′, A′ADD′, D′DCC′, C′CBB′, B′BAA′, the edges AA′, BB′, CC′, DD′ intersect in four points, say AA′, DD′ in α; BB′, CC′ in β; CC′, DD′ in γ; AA′, BB′ in δ: that is, starting with the duad of lines $\alpha\beta$, $\gamma\delta$, the four edges AA′, BB′, CC′, DD′ are the lines $\alpha\delta$, $\beta\delta$, $\beta\gamma$, $\alpha\gamma$ which join the extremities of these duads. Similarly, the four edges AB, CD, A′B′, C′D′ are the lines joining the extremities of a duad; and the four edges AD, BC, A′D′, B′C′ are the lines joining the extremities of a duad. The question arises, "Given two duads, is it possible to place them in space so that the two tetrads of joining lines may be eight of the twelve edges of a hexahedron?" [The duad $\alpha\beta$, $\gamma\delta$ is considered to be given when there is given the tetrahedron $\alpha\beta\gamma\delta$ which determines the relative position of the two finite lines $\alpha\beta$ and $\gamma\delta$.]

Solution by H. J. WOODALL; Professor ZERR; *and others.*



Because $\alpha'\delta'$ is in plane $\alpha\beta\gamma$, and $\alpha'\gamma'$ is in plane $\alpha\beta\delta$, therefore α' must be in line $\alpha\beta$; similarly β' must be in line $\alpha\beta$; therefore $\alpha'\beta'$ is in same straight line as $\alpha\beta$.

Suppose now that a certain position is taken up by $\alpha'\beta'$ (in line with $\alpha\beta$); therefore γ' and δ' will describe circles; but δ' is in plane $\alpha\beta\gamma$, and γ' is in plane $\alpha\beta\delta$; therefore produce these planes to cut the said circles in δ_1 and δ_2, γ_1 and γ_2 respectively; join $\delta_1\gamma_1$, $\delta_1\gamma_2$, $\delta_2\gamma_1$, $\delta_2\gamma_2$: these four lines are equal two and two, and therefore only two different lengths; but by the nature of a duad $\delta'\gamma'$ is given in magnitude; and, since the cutting planes always pass through the centres of these circles and the angle between the planes is constant; therefore it is not always possible to place two duads so as to be the eight edges of twelve of a hexahedron.

10777. (Rev. T. ROACH, M.A.)—A galley rowed by slaves is moored at a point A which is four miles from B the nearest point of the coast, and C

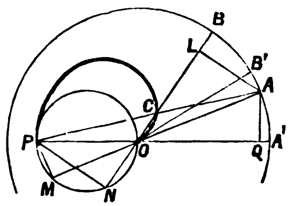
the nearest point of neutral territory is thirty miles from B, the coast-line BC being at right angles to AB. A slave on board the galley can swim, with the aid of the tide, $4\frac{1}{2}$ miles per hour, and can row $7\frac{1}{2}$ miles per hour. Find the point K in BC to which he must swim to reach C in the shortest time. Also, if he was missed after half-an-hour, and his pursuers rowed to B at the rate of four miles an hour, and then after some delay followed him on horseback at the rate of ten miles an hour, find how long they were delayed at B if they arrived at C forty seconds late.

Solution by D. Biddle; Professor Hendricks; *and others.*

Let $x = \text{BK}$. Then $(4^2+x^2)^{\frac{1}{2}}/(4\frac{1}{2}) + (30-x)/(7\frac{1}{2})$ must be a minimum. By reference to the differential coefficients of the two terms, this is the case when $15x/(16+x^2)^{\frac{1}{2}} = 9$, that is, when $x = 3$. The times occupied by the fugitive slave in swimming and rowing respectively will then be $1\frac{1}{9} + 3\frac{9}{15}, = 4\frac{32}{45}$ hours in all, or 4 h. 42′ 40″. His pursuers would get to B an hour and a half after he started from A, and would occupy three hours in riding to C. Consequently, the delay at B must have been $12'\,40'' + 40'' = 13'\,20''$.

10744. (J. J. Barniville.)—O is the centre of a circle, P a fixed point within it, AB an arc of constant magnitude; PA, OB intersect within the circle at C. Find the locus of C as AB moves along the circumference.

Solution by D. Biddle; Rev. J. L. Kitchin, M.A.; *and others.*



Let $\quad \text{AOA}' = \theta,\ \text{AOB} = \alpha,$

$\text{OA} = \text{unity}, \quad \text{OP} = l.$

Then $l\sin\theta/\{(1+l\cos\theta)^2+l^2\sin^2\theta\}^{\frac{1}{2}}$
$= \sin \text{PAO},$

and $(1+l\cos\theta)/\{(1+l\cos\theta)^2+l^2\sin^2\theta\}$
$= \cos \text{PAO}.$

Moreover

$$\{l\sin\theta\cos\alpha+(1+l\cos\theta)\sin\alpha\}/\{(1+l\cos\theta)^2+l^2\sin^2\theta\}^{\frac{1}{2}} = \sin(\text{PAO}+\alpha),$$

and $\qquad l\sin\theta\cos\alpha+(1+l\cos\theta)\sin\alpha : 1 = l\sin\theta : \text{OC}\ (\equiv r);$

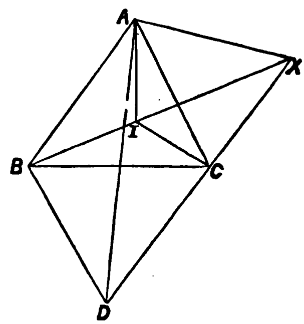
therefore $\qquad r = l\sin\theta/\{\sin\alpha+l\sin(\theta+\alpha)\}.$

The curve is related to the cardioid, and OB′ is the tangent at the origin.

10759. (Professor Décamps.)—Par les sommets B et C d'un triangle ABC, on mène deux droites BI, CI faisant respectivement avec les côté.

AB, AC les mêmes angles que ces côtés font avec la médiane issue de A Démontrer que $AI^2 = BI \cdot CI$.

Solution by Professor GENESE, M.A.



Complete the parallelogram CABD, and let the circle ABD meet DC at X; then BX, AD are equally inclined to AB, hence BX passes through I; also

$$\angle AXI = ADB = DAC = ACI;$$

therefore A, X, C, I are concyclic.

Thus $\angle IAC = IXC = ABI$,

$\angle ICA = DAC = BAI$;

hence the triangles BIA, AIC are similar, $BI : IA = IA : IC$, &c.

[AI is a symmedian, and BA touches the circle IAXC; hence

$$BA^2 = BI \cdot BX = BI \cdot AD,$$

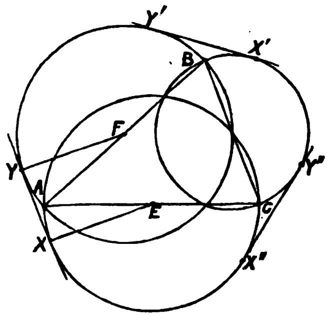
$$CA^2 = CI \cdot AD,$$

$$BA \cdot CA = AI \cdot AD;$$

therefore I is the mid-point of the symmedian chord of the circumcircle.]

10634. (Professor RUSSO.)—Sur chaque côté d'un triangle, comme diamètre, on décrit une circonférence; on mène les tangentes communes à ces circonférences prises deux à deux et on les limite à leurs points de contact. Démontrer que le produit de ces trois tangentes est égal à $(p-a)(p-b)(p-c)$, a, b, c désignant les côtés du triangle, p son demi-périmètre.

Solution by R. H. W. WHAPHAM; Rev. J. L. KITCHIN; *and others.*



Let X, Y; X', Y'; X'', Y'' be the points of contact of the common tangents; then

$$EF^2 = EA^2 + FA^2 - 2EA \cdot FA \cos A;$$

$$\therefore \quad XY^2 = EF^2 - (EX - FY)^2$$

$$= EF^2 - (EA - FA)^2$$

$$= 2EA \cdot FA\,(1 - \cos A)$$

$$= bc \sin^2 \tfrac{1}{2}A.$$

Since $X'Y'^2 = ca \sin^2 \tfrac{1}{2}B$,

$X''Y''^2 = ab \sin^2 \tfrac{1}{2}C$,

$$\therefore \quad XY \cdot X'Y' \cdot X''Y''$$

$$= abc \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$$

$$= (p-a)(p-b)(p-c).$$

[If t_1, t_2, t_3 be the lengths of the tangents, we have

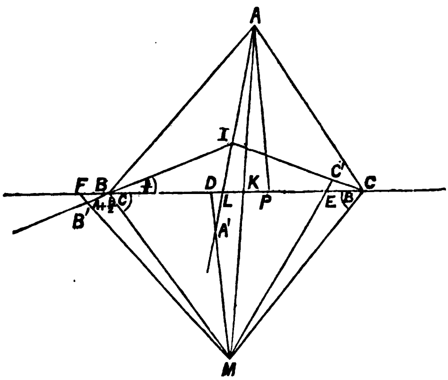
$$t_1^2 = a^2/4-(b-c)^2/4\,; \quad \therefore\; 4t_1^2 = (a-b+c)(a+b-c) = 4(p-b)(p-c).$$

$$\therefore\; t_1^2 t_2^2 t_3^2 = (p-a)^2(p-b)^2(p-c)^2, \text{ or } t_1t_2t_3 = (p-a)(p-b)(p-c).]$$

10565. (Professor BERNES.)—On considère un triangle quelconque ABC, le centre I du cercle inscrit et le point M symétrique de A, relativement au milieu de BC. Par M, on trace MD perpendiculaire à BC et ME, MF faisant, avec BC, deux angles égaux à l'angle A, l'un de même sens que l'angle de AB avec AC, l'autre de même sens que l'angle de AC avec AB. Ces trois droites rencontrent respectivement IA, IB, IC en A′, B′, C′. Démontrer que $MA' = a \tan \frac{1}{2}A$, $MB' = b$, $MC' = c$. On fera voir aussi que, si E′, F′ sont les rencontres de ME et MF avec IA, on a la relation $1/ME' + 1/MF' = 1/R \cot \frac{1}{2}A$; où R est le rayon du cercle circonscrit à ABC, et où ME′ et MF′ doivent être affectés de signes selon le sens de AI où tombent E′ and F′.

Solution by W. J. GREENSTREET, M.A. ; H. J. WOODALL ; *and others.*

Let P be the foot of the perpendicular from A on BC. K is mid-point of BC, AM ; therefore BMCA is a parallelogram, and MB = AC, MC = AB.



$$MBB' = A + \tfrac{1}{2}B = MB'B, \quad \text{similarly} \quad MCC' = MC'C = B + \tfrac{1}{2}C;$$

therefore $\qquad MB' = MB = AC,\; MC' = MC = AB.$

Again,

$$MA' = MD - A'D = c \sin B - \frac{AP}{PL}\,.\,DL = b \sin C - c \sin B \left(\frac{BL-BD}{BP-B}\right),$$

and $\qquad BL = c \sin \frac{1}{2}A / \sin(B + \frac{1}{2}A),\quad BD = b \cos C,\quad BP = c \cos B;$

therefore $\quad \mathrm{MA}' = b\sin \mathrm{C} - c\sin \mathrm{B}\left[\dfrac{c\sin\frac{1}{2}\mathrm{A} - b\cos \mathrm{C}\sin(\mathrm{B}+\frac{1}{2}\mathrm{A})}{c\cos \mathrm{B}\sin(\mathrm{B}+\frac{1}{2}\mathrm{A}) - c\sin\frac{1}{2}\mathrm{A}}\right]$

$$= b\sin \mathrm{C}\left[\frac{a\sin(\mathrm{B}+\frac{1}{2}\mathrm{A}) - 2c\sin\frac{1}{2}\mathrm{A}}{c\cos \mathrm{B}\sin(\mathrm{B}+\frac{1}{2}\mathrm{A}) - c\sin\frac{1}{2}\mathrm{A}}\right]$$

because $\quad b\cos \mathrm{C} + c\cos \mathrm{B} = a$

$$= \sin \mathrm{B}\left[a\sin(\mathrm{B}+\tfrac{1}{2}\mathrm{A}) - 2a\sin\tfrac{1}{2}\mathrm{A}\,\frac{\sin \mathrm{C}}{\sin \mathrm{A}}\right] \Big/ \left[\cos(\mathrm{B}+\tfrac{1}{2}\mathrm{A})\sin \mathrm{B}\right]$$

because $\quad \sin\frac{1}{2}\mathrm{A} = \sin\left[(\mathrm{B}+\frac{1}{2}\mathrm{A}) - \mathrm{B}\right]$, &c.

$$= a\,\frac{\sin(\mathrm{B}+\frac{1}{2}\mathrm{A})\cos\frac{1}{2}\mathrm{A} - \sin(\mathrm{A}+\mathrm{B})}{\cos(\mathrm{B}+\frac{1}{2}\mathrm{A})\cos\frac{1}{2}\mathrm{A}}$$

$$= -a\,.\cos(\mathrm{B}+\tfrac{1}{2}\mathrm{A})\sin\tfrac{1}{2}\mathrm{A}/\left[\cos(\mathrm{B}+\tfrac{1}{2}\mathrm{A})\right]\cos\tfrac{1}{2}\mathrm{A}$$

$$= -a\tan\tfrac{1}{2}\mathrm{A}.$$

Also $\quad \mathrm{ME}' = \mathrm{MA}'\cos(\mathrm{B}+\frac{1}{2}\mathrm{A})/\sin(\mathrm{B}+\frac{3}{2}\mathrm{A})$,

and $\quad \mathrm{MF}' = \mathrm{MA}'\,.\cos(\mathrm{B}+\frac{1}{2}\mathrm{A})/\sin(\mathrm{B}-\frac{1}{2}\mathrm{A})$;

therefore algebraical sum of ME' and MF'

$$= \left\{\sin(\mathrm{B}+\tfrac{3}{2}\mathrm{A}) \sim \sin(\mathrm{B}-\tfrac{1}{2}\mathrm{A})\right\}/a\tan\tfrac{1}{2}\mathrm{A}\cos(\mathrm{B}+\tfrac{1}{2}\mathrm{A})$$

$$= \frac{2\sin \mathrm{A}}{a\tan \mathrm{A}} = (1/\mathrm{R})\cot \mathrm{A} \text{ or } 1/(\mathrm{R}\tan \mathrm{A}).$$

10567. (Professor Wolstenholme, M.A., Sc.D.)—At the point P $(am^2, 2am)$ of the parabola $y^2 = 4ax$ is drawn the circle of curvature to the parabola, and the remaining common tangent to the circle of curvature and parabola touches them respectively in P′, P″; prove that

$$(1)\ \mathrm{P'P''} = 16am^3(1+m^2)^{\frac{3}{2}}, \quad \mathrm{PP'} = 8am(1+m^2)^{\frac{3}{2}}/(1+4m^2)^{\frac{1}{2}};$$

$$(2)\ \mathrm{PP''} = 8am(1+m^2)^{\frac{3}{2}}(1+4m^4)^{\frac{1}{2}};$$

(3) if Q be the remaining common point of the circle and parabola, the tangent to the circle at Q will make with the directrix an angle three times that which the tangent at P does, and the chord of the parabola along this tangent $= 2\mathrm{P'P''}/(1-3m^2)$; and (4) if R, T be the points of intersection of P′P″ with PQ, and the tangent at P, respectively,

$$\mathrm{TP} : \mathrm{TP''} = \mathrm{P'R} : \mathrm{RP''} = 1 : 1+4m^2.$$

Solution by Rev. T. Galliers, M.A.; R. Knowles, B.A.; *and others.*

(1), (2). The equation to the common tangent is

$$x + m(4m^2+3)\,y + am^2(4m^2+3)^2 = 0;$$

the coordinates of P′, P″ are $\quad 3am^2(4m^2+3)/(1+4m^2)$,

$$-2am(8m^4+8m^2+3)/(1+4m^2);\ am^2(4m^2+3)^2,\ -2am(4m^2+3);$$

and from these P′P″, PP′, PP″ are found as above.

(3) The equation to the tangent to the circle at Q is

$$(3m^2-1)\,x+m\,(m^2-3)\,y-3am^2(7m^2+3) = 0 \;\ldots\ldots\ldots\ldots\ldots (\alpha),$$

and that at P, $$x-my+am^2 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta);$$

if (β) makes an angle θ with the directrix,

$\theta = \tan^{-1} m$; $3\theta = \tan^{-1} m\,(m^2-3)/(3m^2-1) =$ the angle (α) makes with it.

The coordinates of the points where (α) meets the parabola are

$$9am^2,\ -6am;\ am^2(7m^2+3)^2/(3m^2-1)^2,\ 2am\,(7m^2+3)/(3m^2-1);$$

and the distance $= 32am^3(1+m^2)^{\frac{3}{2}}/(3m^2-1)^2 = 2\text{P}'\text{P}''/(3m^2-1)^2$.

(4) The coordinates of R, T, and the lengths TP, TP″, P′R, RP″ are respectively

$$am^2(4m^2+3)(2m^2+3)/(2m^2+1),\ -2am\,(4m^4+6m^2+3)/(1+2m^2);$$

$$-am^2(4m^2+3),\ -2am\,(1+2m^2);\ 4am\,(1+m^2)^{\frac{3}{2}};\ 4am\,(1+m^2)^{\frac{3}{2}}(1+4m^2);$$

$$8am^3(1+m^2)^2(1+4m^2)/(1+2m^2);\ 8am^3(1+m^2)^2(1+4m^2)^2/(1+2m^2);$$

and the ratios are as stated.

10681. (R. Tucker, M.A.)—Two parabolas intersect in four concyclic points, O is the centre of the circle, and K the intersection of the axes; S, L, S′, L′ are the respective foci and latera recta of the curves. Prove that (1) $4\text{OK}^2 = \text{L}^2+\text{L}'^2$; (2) the square on radius of the circle $= \text{L}\,.\,\text{SN}+\text{L}'\,.\,\text{S}'\text{N}'$ (N, N′ being the projections of O on the axes); (3) the polar of either vertex with respect to the other curve meets its polar with respect to the circle on the tangent through that vertex, and the curve cuts the tangent midway between the vertex and that point of section; and (4) the rectangles under the intercepts from the vertices made on the respective axes by the curves and the circle are respectively equal.

Solution by the Proposer.

Since the two parabolas have four concyclic points in common, their axes are at right angles, and their equations may be taken to be

$$(\alpha)\ y^2 = 4mx,\quad (\beta)\ ax^2+2gx+2fy+c = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)},$$

and that of the circle, $$a(x^2+y^2)+2(g-2am)\,x+2fy+c = 0 \;\ldots\ldots\ldots \text{(ii.)}.$$

The vertex of (β) is $\quad -g/a,\ \ (g^2-ac)/2fa$;

therefore equation to axis is $x+g/a = 0$, and $\text{L}' = -2f/a$.

Coordinates of centre O of (ii.) are $(2am-g)/a,\ -f/a$; therefore distances of O from axes of (α) to (β) are $\frac{1}{2}\text{L}'$, $\frac{1}{2}\text{L}$, $\therefore$ (1) $4\text{OK}^2 = \text{L}^2+\text{L}'^2$.

The $(\text{radius})^2$ of (ii.) $= (4a^2m^2-4amg+g^2+f^2-ac)/a^2 = \rho^2$,

and $$\text{SN} = -m+(2am-g)/a = (am-g)/a,$$

and $$\text{S}'\text{N}' = (ac-g^2)/2fa-f/2a = -(g^2+f^2-ac)/2fa;$$

therefore $$\rho^2 = \text{L}\,.\,\text{SN}+\text{L}'\,.\,\text{S}'\text{N}' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The polars of (x', y') with regard to (β) and (ii.) are

$$axx' + g(x+x') + f(y+y') + c = 0,$$

$$axx' + ayy' + (g-2am)(x+x') + f(y+y') + c = 0;$$

and of vertex of (i.)

$$gx + fy + c = 0, \quad (g-2am)x + fy + c = 0 \;\ldots\ldots\ldots\ldots\ldots \text{(iii.)}.$$

These evidently pass through the same point $(0, -c/f)$ on the tangent at vertex of (α); the like result is readily *proved* for (β) or can be inferred by analogy. (3), (β) cuts the tangent at vertex of (α) in $(0, -c/2f)$; hence, &c.

From (ii.), (β), intercepts on axis of (α) from vertex are given by the equations $\quad ax^2 + 2(g-2am)x + c = 0, \quad ax^2 + 2gx + c = 0,$
whence (4) is obvious. It will be seen that SN, S'N' are equal to the focal vectors of the points where axes of (β), (α) cut (α), (β) respectively.

10417. (Professor ORCHARD, M.A.)—A series of right-angled isosceles triangles, each of base a, are placed continuously along the positive side of the axis of x. Show how to express the locus consisting of their sides as a periodic function of x.

Solution by D. BIDDLE.

The zigzag deflections occur at uniform intervals, in x, of $\frac{1}{2}a$. Let $x = n(\frac{1}{2}a) + z$; then we have

$$y = \tfrac{1}{4}a\left\{1-(-1)^n\right\} - z(-1)^{n+1}$$

$$= \tfrac{1}{4}a\left\{1-(-1)^n\right\} - \left\{x - n(\tfrac{1}{2}a)\right\}(-1)^{n+1};$$

$$\int_0^{(n+k)\frac{1}{2}a} f(x)\,dx = n\int_0^{\frac{1}{2}a} z\,dz + \int_0^{k\frac{1}{2}a}\left[\tfrac{1}{4}a\left\{1-(-1)^n\right\} - z(-1)^{n+1}\right]dz$$

$$= \tfrac{1}{8}a^2\left[n + k\left\{1-(-1)^n\right\} - k^2(-1)^{n+1}\right].$$

10741. (WALTER STOTT.)—Prove that the solution of

$$\frac{d^2y}{dx^2} - ax\frac{dy}{dx} + \mu ay = 0$$

is $$y = x^\mu - \frac{\mu(\mu-1)}{2a}x^{\mu-2} + \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{2a\,.\,4a} - \ldots.$$

Solution by Prof. SIRCOM; Rev. J. L. KITCHIN; *and others.*

Putting $y = \Sigma(a_n x^n)$, we have for the relation amongst the coefficients

$$n(n-1)a_n = (n-2-\mu)\,a\,a_{n-2},$$

whence, a_0 and a_1 being arbitrary, the complete solution is

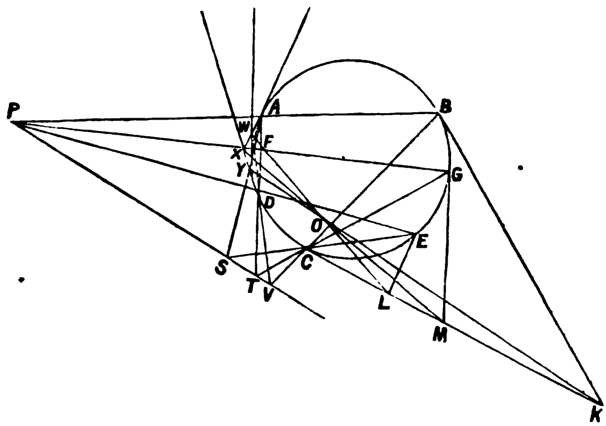
$$y = a_0\left(1-\frac{\mu}{2!}ax^2+\frac{\mu(\mu-2)}{4!}a^2x^4-\dots\right)$$
$$+a_1x\left(1-\frac{\mu-1}{3!}ax^2+\frac{(\mu-1)(\mu-3)}{5!}a^2x^4-\dots\right).$$

If μ is a positive integer, the first or second series ends with the term in x^μ according as μ is even or odd.

6140. (S. A. Renshaw.)—If PAB, PFG, PDE be any three secants cutting a circle in the points A, B; F, G; D, E, respectively; and if tangents be drawn at the points A, F, D meeting one another in W, X, Y, and also at B, E meeting the tangent at any point C in K, L, M respectively; prove that KY, MX, LW pass through the same point O.

Solution by the Proposer.

Completing the figure, we have. by a theorem already proved, the intersections of AF, EC; AD, GC; FD, BC in straight line passing



through P; but, since K and Y are the poles of BC and FD, M and X those of C′ and C, and AD and L and W those of CE and AF, therefore the three lines KY, MX, LY all pass through the same point, viz., the pole of the line PV.

10713. (W. J. Greenstreet, M.A.)—In a circle the radii OA, OB are at right angles. Describe a circle on OB as diameter. Prove that, if the join of A and the centre of the smaller circle meet the latter in D,

AD is a side of the regular in-decagon; and the chord of the larger circle tangential to the smaller is a side of the regular in-pentagon.

Solution by R. Knowles, M.A.; Prof. Zerr; *and others.*

Take OA, OB for axes; O′ centre of circle in OB; the equations to the circles are $x^2+y^2=c^2,\quad x^2+y^2-cx=0$ ....(1, 2);
$AO'^2=5c^2/4$; $\therefore$ $AD=AO'-O'D=(5^{\frac{1}{2}}-1)\,c/2$ = side of in-decagon;
if $hx+ky=c^2$ be a chord of (O) which is tangential to (O′), $h^2+k^2=(h-2c)^2$; and the condition that this = side of in-pentagon is

$$(h-2c)^2=2c^2(3-5^{\frac{1}{2}});\quad \text{therefore } h=(3-5^{\frac{1}{2}})\,c \text{ or } (1+5^{\frac{1}{2}})\,c.$$

The latter value makes k impossible; therefore there is only one real chord of (O), tangential to (O′), which = side of in-pentagon.

10707. (J. D. H. Dickson, M.A.)—A common pendulum is allowed to swing from rest in a horizontal condition, and at any point of the motion the string is cut. If the pendulum, of length c, be suspended from the origin, and the axes of x and y be horizontal and vertical, prove that the locus of the focus of the path of the bob of the pendulum in its subsequent parabolic motion is

$$4(x^2+y^2-c^2)^3=27c^4y^2.$$

Solution by Professors Anderson, Sarkar, *and others.*

Let the initial position of the string be the production of the axis of x in the negative direction; and let the angle turned through before the string is cut be θ; then the coordinates of the point of projection are $(-c\cos\theta,\ -c\sin\theta)$, the angle of projection $\theta-\frac{1}{2}\pi$, and the velocity of projection $(2cg\sin\theta)^{\frac{1}{2}}$. The coordinates of the focus are

$$x=-2c\sin^2\theta\cos\theta-c\cos\theta,\quad y=-2c\sin^3\theta;$$

therefore $x^2+y^2-c^2=3c^2\sin^2\theta$, whence the result follows.

10575. (W. J. C. Sharp, M.A.)—Prove that the twelve centres of similitude of the four circles which touch the three sides of a triangle are the angular points, each taken twice, and the feet of the external and internal bisectors of the angles.

Solution by H. J. Woodall; Morgan Brierley; *and others.*

Let the triangle be ABC, and call the in-circle O, and the ex-circles A, B, C respectively; then

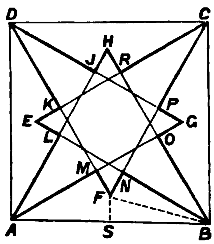
External centres of O and A, B, C respectively are A, B, C respectively;
Internal ,, B and C, C and A, A and B ,, ,, A, B, C ,,

Internal centre of O and A lies on the join of centres, *i.e.* the in-bisector of A,
„ „ „ tangent line BC,
„ O and A is the foot of the internal bisector of A,
„ O and B „ „ B,
„ O and C „ „ C,

External centre of B and C lies on the join of the centre, *i.e.* the external bisector of A, and therefore at the foot of this external bisector;
external centre of C and A „ „ the „ B;
„ „ A and B „ „ „ „ C;
therefore, &c.

10166. (C. A. Swift.)—Equilateral triangles are described on the four sides of a square, the triangles all lying within the square. Show that the area of the eight-pointed star-shaped figure formed by the vertices of the triangle and the corners of the square, together with three times the area of the square, is equal to eight times the area of one of the equilateral triangles.

Solution by Professor Putnam, M.A.



Let ABCD be a square, its side $= 2a$. Describe the equilateral triangles and there will be formed the star-shaped figure AMFNBOGPC, &c., having eight "points" A, B, C, and D, at the corners of the square, and E, F, G, and H at the vertices of the triangles.

The area of the square not covered by this star consists of four equal figures, one of which is AMFNBA.

Draw the perpendicular FS and also FB. The right-angled triangles FSB and FNB are equal. $\text{FS} = \text{FN} = 2a - a\sqrt{3}$, $\text{SB} = \text{NB} = a$. Area $\text{FNB} = \frac{1}{2}a\,(2a - a\sqrt{3})$. Area of figure AMFNBA = four times $\text{FNB} = 2a\,(2a - a\sqrt{3})$ and total area not covered by star $= 8a\,(2a - a\sqrt{3})$.

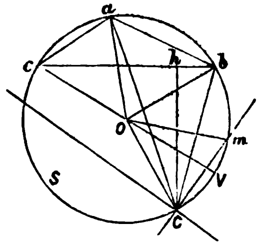
Star $= 4a^2 - 8a\,(2a - a\sqrt{3}) = 8a^2\sqrt{3} - 12a^2$.

Star + 3 times square $= 8a^2\sqrt{3} = 8$ times area equilateral triangle.

[To the solution given on p. 49 of Vol. LII., Prof. Putnam offers the following objections:—(1) The star-shaped figure there shown is not formed by the construction indicated in the problem in any line, but by joining the points of the star. This in a regular five-pointed star would form a pentagon. (2) The figure is not eight-pointed, but four-pointed, the re-entrant angle not being a point as the word is used in speaking of stars. (3) The figure is not star-shaped, and is in fact not a figure, when the square PRQS is "cut out" of it, though the areas of the four disconnected equal triangles do equal the area contemplated in the problem.]

10177. (F. R. J. HERVEY.)—Prove the following statements relating to the system of rectangular hyperbolas passing through four given mutually orthocentric points:—(1) The mean point of the centres of any six *equal* hyperbolas is the mean of the conjugate triad determined by the given points. (2) The asymptotes of six such hyperbolas cut the locus of centres again in two sets of six points, diametrically opposite each to each, such that the mean point of each set is the mean of the given points. (3) The system of mean positions on the circle determined by the six points in (1) is independent of the magnitude of the hyperbolas; the same applies to either set in (2). (4) The three *maximum* hyperbolas have the property that their normals at the given points are concurrent; the mean point of their centres divides in the ratio 1 : 2, the distance between the means in (2) and (1).

Solution by the PROPOSER.



Let S (centre O, radius unity) be the locus of the centre C of the variable hyperbola, m the other intersection of either asymptote with S; the angular velocities of the asymptotes and of OC and Om are as $1 : 4 : -2$. Let abc be the conjugate triad corresponding to the given tetrad, V a vertex of the hypocycloidal envelope of the asymptotes, 2ϕ the angle VOC, and $2\theta_1$, $2\theta_2$, $2\theta_3$ the angles VOa, VOb, VOc, so measured that we may have (1) $\theta_1+\theta_2+\theta_3 = 0$, a relation which is obvious when a, b, c are the vertices of the cycloid, and follows for any other position (with the given envelope) by considering the above velocity-ratio of the line-pairs bisecting the angles of abc.

Let the equation of the hyperbola referred to asymptotes be $2xy = \mu$, and draw Ch perpendicular to bc: since a is the pole of bc, we have (without reference to sign) Ca . Ch = μ. But

$$Ca = 2\sin\tfrac{1}{2}COa,\quad Ch = Cb\sin Cbc = 2\sin\tfrac{1}{2}COb\sin\tfrac{1}{2}COc,$$

whence (2) $\qquad -\mu = 4\sin(\phi-\theta_1)\sin(\phi-\theta_2)\sin(\phi-\theta_3),$

the negative sign being assumed for the sake of agreement with p. 70, Vol. LII. Now, it is evident that each time C passes through a vertex of abc, the hyperbola passes into the alternate quadrants, and μ should change sign, as above. The variable hyperbola, thus considered, requires a *double* revolution of its centre to pass through all its phases.

Let $\varpi \equiv \cos\phi + i\sin\phi$, $\tau_1 \equiv \cos\theta_1 + i\sin\theta_1$, &c.; $\tau_1\tau_2\tau_3 = 1$, by (1). Hence, if $3\alpha = \tau_1^2+\tau_2^2+\tau_3^2$, $3\beta = \tau_1^{-2}+\tau_2^{-2}+\tau_3^{-2}$,

(2) becomes $\qquad \varpi^6 - 3\alpha\varpi^4 - 2\mu i\varpi^3 + 3\beta\varpi^2 - 1 = 0$ ....(3),

For a given value of μ, the squares of the roots of this equation are the vectors (or their ratios to OV) from O to the centres of the corresponding hyperbolas; their mean is α, the vector to the mean point of abc.

2. The roots themselves are the vectors, reflected with respect to OV, of the six positions of the intersection m of that asymptote which touches

S at V when $\phi = 0$. Now, S is divided into three arcs by the points of the triad, and into six by the mid-points of the tetrad; and, while C describes one of the former arcs, m describes one of the latter. It is hence evident that the six points m lie in pairs in *alternate* arcs of the above six.

3. For n directions in a plane, there are n other directions, each of which is equally entitled to be called a mean of the first set; and similarly for points on a circle. From the product of roots in (3), it follows that the mean points are, for the points C, the vertices of the cycloid and their opposites; and the same, turned through $\pi/6$, for the points m.

4. For the theory of what follows, I must refer to Question 9525, Vol. LII. Equation (2), above, supposing ϕ and μ constant, and θ_1, &c. variable, may be regarded as an equation to the corresponding hyperbola, the angular coordinates θ determining a tetrad on the curve. [It may be reduced to the form (2) of 9525, with which it is made identical by assuming $\xi + i\eta = -3a$: this gives the relation between P and p there stated.] Let CD be a diameter of S. That a hyperbola, centre D, may have concurrent normals, the hyperbola, centre C, must pass through D, that is, through the tetrad having two coincident points at D; or, its equation must be satisfied by $2\theta_1 = 2\theta_2 = -\theta_3 = 2\phi - \pi$, which gives $\mu = 4 \sin 3\phi$. [In fact, the form of (2), 9525, required is that of the *normal* to the cycloid; as may be otherwise inferred.] Now, for a maximum hyperbola, μ must be a maximum or minimum, and the condition $d\mu/d\phi = 0$, from (2), will be found to have the form just obtained, with $\phi + \pi/2$ in place of ϕ. The vectors to the centres of the maximum hyperbolas are the values of ϖ^2 satisfying $\varpi^6 - a\varpi^4 - \beta\varpi^2 + 1 = 0$. [See also the Solution of Quest. 9525, on p. 69 of Vol. LII.]

10418. (Professor SYAMADAS MUKHOPÂDHYÂY.)—ABC is a triangle having the angles at A not less than 120°. P is any point. Prove that the arithmetical sum of PA, PB, PC is least when P is taken at A.

Solution by Professors ANDERSON, BEYENS, *and others.*

The geometrical construction for finding the point P which makes the expression PA + PB + PC a minimum, where A, B, C are the corners of a triangle and P in its plane, is to describe segments of circles on the sides containing angles of 120°; their common point is the point required. If the angle A is 120°, the circles intersect at A and it itself is the required point. If A is > 120°, we can show that it still makes the sum a minimum. Describe an ellipse passing through A with foci B and C. Along the tangent of this ellipse at A the sum PA + PB + PC plainly increases for both directions. Now consider the variation along the normal, and take P_1 a point very near A on the normal drawn inwards, and P_2 similarly on the normal drawn outwards. Then

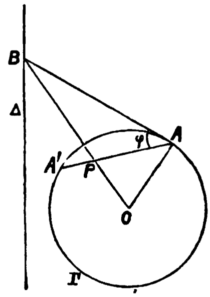
$$P_2A + P_2B + P_2C = b + c + P_2A\,(1 + 2\cos\tfrac{1}{2}A) > b + c,$$

and $P_1A + P_1B + P_1C = b + c + P_1A\,(1 - 2\cos\tfrac{1}{2}A) > b + c$ if $\tfrac{1}{2}A > 60°$.

Thus the sum is a minimum for A.

10702. (Professor TARRY.)—On donne une droite, une conique Γ, et deux divisions homographiques sur cette conique. Les tangentes à la conique, en deux points homologues A, A′, rencontrent la droite en deux points; les droites qui joignent ces deux derniers points au pôle de la droite des points doubles, coupent la corde AA′ en des points dont le lieu est une conique. Quand la droite donnée est tangente à Γ, le lieu est un système de deux droites.

Solution by Professors SCHOUTE, BEYENS, *and others.*



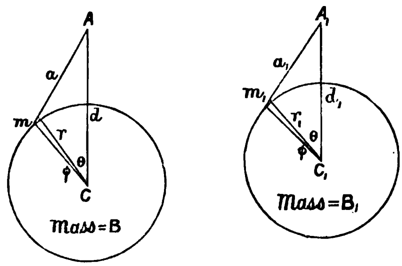
Projetons la figure, de manière que les deux points doubles se projettent à l'infini dans les deux points cycliques. Cela transforme le problème à démontrer dans le suivant:

"On donne une circonférence Γ et une droite Δ. Trouver le lieu du sommet P d'un triangle ABP, dont A et B se meuvent sur Γ et Δ, tandis que AB touche Γ, que BP passe par le centre O de Γ, et que l'angle A reste constant."

Ce lieu se compose des droites isotropes par O et d'une conique. En effet, l'ordre de la partie essentielle du lieu se trouve en cherchant les points P situés sur Δ (les deux points communs à Γ et à Δ), ou les points P situés sur la circonférence, enveloppe de AA′ (les quatre points, pour lesquels OB = OA sec ϕ), &c.

10668. (R. CHARTRES, M.A.)—Show that a slight variation in NEWTON's method of proof (Book I., Prop. 71) will give, in a very elementary manner, the attraction of a spherical shell on an external particle.

Solution by the PROPOSER.



NEWTON, in this famous proposition, uses two equal spherical surfaces; but in the next proposition he employs two homogeneous spheres of equal density, whose radii are in the ratio of the distances from the centres to the external points.

Had he adopted this method in Prop. 71, the proof would be very elementary, and the whole attraction obtained. Let the thickness of the shells be equal, and indefinitely small, and let m, m_1 be two indefinitely small portions similar and similarly placed.

By properly taking out units, the direct attaction of (m) on $A = m/a^2$, and of (m_1) on $A_1 = m_1/a_1^2$; but $m : m_1 = r_2 : r_1^2$ and $a : a_1 = r : r_1$ (similar triangles); therefore direct attraction of (m) on A = that of (m_1) on A_1, and their resolved attractions towards the centre will be equal. Hence, attraction of shell B on A = that of B_1 on A_1 = constant; but B/r^2 = constant, and therefore B/d^2 = constant; therefore attraction of B on A varies as B/d^2, say $= k\,(B/d^2)$, which proves the proposition. Now A and C remaining fixed, let (r) become indefinitely small, and the shell a particle of mass (n); therefore attraction of $(n) = k\,(n/d^2)$; but this attraction is (n/d^2); therefore $k = 1$. Therefore attraction of B on $A = B/d^2$.

10612. (J. J. BARNIVILLE.)—Prove that, when $\beta > \alpha + 1$,

$$\frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \dots = \frac{\alpha}{\beta-\alpha-1},$$

$$(\alpha+\beta+1)\frac{\alpha^2}{\beta^2} + \dots + (\alpha+\beta+5)\frac{\alpha^2(\alpha+1)^2(\alpha+2)^2}{\beta^2(\beta+1)^2(\beta+2)^2} + \dots = \frac{\alpha^2}{\beta-\alpha-1}.$$

Solution by R. H. W. WHAPHAM, B.A.; H. J. WOODALL; *and others.*

1. Let $u_n = \dfrac{\alpha(\alpha+1)\dots(\alpha+n-1)}{\beta(\beta+1)\dots(\beta+n-1)}$, $v_n = \dfrac{\alpha(\alpha+1)\dots(\alpha+n-1)(\alpha+n)}{\beta(\beta+1)\dots(\beta+n-1)}$;

$$\therefore\ v_n - v_{n-1} = \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\dots(\beta+n-1)}(\alpha+1-\beta) = (\alpha+1-\beta)\,u_n;$$

therefore, letting $n = 1, 2, 3$, and adding, we get

$$\Sigma u = \frac{-v_0}{\alpha+1-\beta} = \frac{\alpha}{\beta-\alpha-1}.$$

2. Let $$u_n = (\alpha+\beta+2n-1)\frac{\alpha^2(\alpha+1)^2\dots(\alpha+n-1)^2}{\beta^2(\beta+1)^2\dots(\beta+n-1)^2},$$

$$v_n = \frac{\alpha^2(\alpha+1)^2\dots(\alpha+n-1)^2(\alpha+n)^2}{\beta^2(\beta+1)^2\dots(\beta+n-1)^2};$$

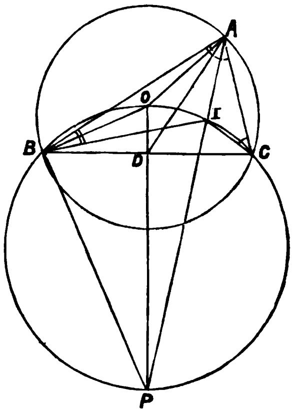
$$\begin{aligned}\therefore\ v_n - v_{n-1} &= \frac{\alpha^2(\alpha+1)^2\dots(\alpha+n-1)^2}{\beta^2(\beta+1)^2\dots(\beta+n-1)^2}(\alpha^2-\beta^2+2\alpha n-2\beta n+2\beta+2n-1)\\ &= \frac{\alpha^2(\alpha+1)^2\dots(\alpha+n-1)^2}{\beta^2(\beta+1)^2\dots(\beta+n-1)^2}(\alpha+\beta+2n-1)(\alpha+1-\beta)\\ &= (\alpha+1-\beta)\,u_n;\end{aligned}$$

therefore, letting $n = 1, 2, 3, \dots$ &c., and adding, we get

$$\Sigma u = \frac{-v_0}{\alpha+1-\beta} = \frac{\alpha^2}{\beta-\alpha-1}.$$

10759 (Professor DECAMPS.)—Par les sommets B et C d'un triangle ABC, on mène deux droites BI, CI faisant respectivement avec les côtés AB, AC les mêmes angles que ces côtés font avec la médiane issue de A. Démontrer que $AI^2 = BI \cdot CI$.

Solution by G. E. CRAWFORD, B.A.; D. BIDDLE; *and others.*



Taking O the circumcentre, join OA, OB, OD, and produce OD to meet the circumcircle round BOC in P. Join AP. cutting this circumcircle in I, and join BI, CI. We first show that BI, CI are the lines required to be drawn, thus:—

$$OD \cdot OP = OB^2 = OA^2;$$

$\therefore \quad \angle OAD = OPA = OBI.$

But $\quad \angle OAB = OBA;$

therefore $\quad \angle BAD = ABI.$

Similarly, $\quad \angle CAD = ACI.$

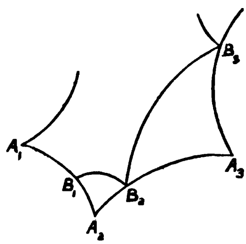
Now $\angle CAI = A - BAI$

$$= BIP - BAI = ABI,$$

similarly $\angle BAI = ACI$; hence AIB, CIA are similar, $\therefore AI^2 = BI \cdot CI$.

10701. (Professor MORLEY.)—Prove that, (1) if a closed curve have an *odd* number of real cusps, any involute will be a closed curve; and (2) if it have an *even* number, any involute will, in general, proceed spirally to infinity.

Solution by H. J. WOODALL; Professor MORLEY; *and others.*



Let $A_1, A_2, A_3 \ldots$ be the cusps; start from B_1 and unwrap in the negative sense; also, let $A_1A_2 = a_1$, $A_2A_3 = a_2$, ... $A_1B_1 = x$; then

$$A_2B_2 = a_1 - x, \quad A_3B_3 = -a_1 + a_2 + x,$$
$$A_4B_4 = a_1 - a_2 + a_3 - x.$$

If four cusps, the series proceeds

$$-a_1 + a_2 - a_3 + a_4 + x, \quad 2a_1 - a_2 + a_3 - a_4 - x,$$
$$-2a_1 + 2a_2 - a_3 + a_4 + x,$$

and can only close if $a_1 + a_3 = a_2 + a_4$.

If three cusps, it goes on, after

$$A_1B_4 = a_1 - a_2 + a_3 - x, \quad A_2B_5 = a_2 - a_3 + x, \quad A_3B_6 = a_3 - x, \quad A_1B_7 = x.$$

Thus after two turns we reach the point of departure. And so generally, for an odd number the series ends after two turns, for an even number it never closes unless the lengths of the sets of alternate arcs are equal.

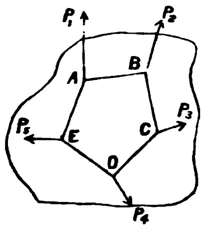
Inflexions increase the number of cusps of the involute, but do not seem to affect the above result. If the curve is convex outwards, like an epicycloid, the involute will proceed to infinity, when the number k is even. Of course it will not in the epicycloid, since there the condition between the lengths of arcs is satisfied.

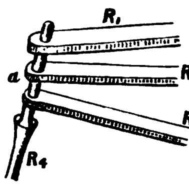
In considering the involute as got by unwrapping a string, what is to be said when the curve consists of distinct ovals? The books do not even say what is to be done for cusps and inflexions.

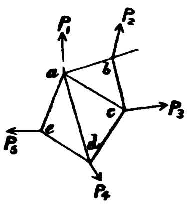
10771. (G. E. Crawford, B.A.)—Assuming the principle of virtual work for a rod in equilibrium under forces at its extremities, prove, without any reference to the six conditions of equilibrium, that the principle must hold for any rigid body.

Solution by F. A. Colebridge; Prof. Chakrivarti; *and others.*

Let any system $P_1, P_2 \ldots$ of forces, applied at points A, B, C, keep a rigid body in equilibrium. Conceive a polygon $abc\ldots$ equal and similar to ABC. Let it consist of rods freely jointed at the angles by pegs, and let additional rods ac, ad ensure its being rigid.







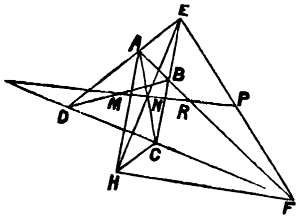
Then forces $P_1, P_2 \ldots$ parallel to the former will keep it in equilibrium. Since the whole is in equilibrium, so is each part; therefore each rod and peg is separately in equilibrium. Therefore in the displacement the equation of virtual work holds for the forces which act on each *peg*. Therefore, if $R_1R_2R_3R_4$ be the reactions of the four rods which meet at a on the peg, and if we denote by ${}^vP$ the virtual work done by any force P, then

$${}^vP_1 + {}^vR_1 + {}^vR_2 + {}^vR_3 + {}^vR_4 = 0.$$

Hence adding all such equations, $\Sigma^v P_1 + \Sigma\, {}^vR_1 = 0$. But the equation of virtual work holds also for the forces which act on each *rod*; and these are only the R's over again, but with opposite signs. Hence adding all such equations, we get $-\Sigma\, {}^vR_1 = 0$. Add this to the former equation, then $\Sigma\, {}^vP_1 = 0$. Hence the proposition holds for the system *abcde*; and therefore it holds for ABCDE.

10796. (Professor GENESE, M.A.)—The join of the mid-points of the diagonals of a quadrilateral in a circle makes the same angles with any side that the third diagonal makes with the opposite side.

Solution by J. ADAMS, M.A.; R. KNOWLES, B.A.; *and others.*



Let ABCD be the cyclic quadrilateral; M, N the mid-points of the diagonals. Draw AH parallel to EC, and CH parallel to EA. Join EH, FH; and let P be the mid-point of EF; therefore NR is parallel to HF; therefore angle NRA = angle HFA. But, because quadrilateral is cyclic,

$$FC : FB = CA : BD = EA : EB;$$

$$\therefore \quad CF : CH = BF : BE;$$

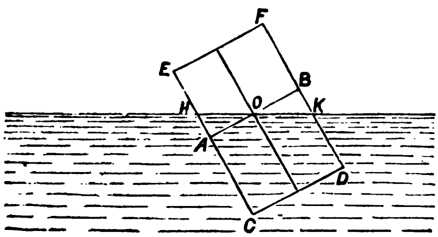
and angle HCF is supplement of HCD, *i.e.* of CDA, and therefore is equal to angle EBF. The triangles HCF, EBF are therefore similar;

hence $\angle$ CFE = HFA = NRA. From this the other cases easily follow.

10813. (J. J. WALKER, F.R.S.)—Show that, in the case of a homogeneous rectangular parallelepiped floating in a liquid (the densities of the two being $\rho : 1$), there will be a position of stable equilibrium in which the longest edge is horizontal, and the other two are inclined, provided the ratio lies between the square roots of $2(1-\rho)(4\rho-1)$ and $6\rho(1-\rho)$: *e.g.*, if $\rho : 1 = 4 : 5$, and the ratio of the least to the mean edge $= \sqrt{9} : \sqrt{10}$, then the inclination of the former to the horizontal, in the position referred to, will be about 20°.

Solution by Professors ZERR, MUKHOPADHYAY, *and others.*

Since the solid is homogeneous and the longest edge horizontal, we need only consider the plane rectangle which forms the face made by the



least and mean sides. We get, for measure of stability and inclination to horizontal, $S = P\left(\frac{b^3}{12A} \pm e\right)\theta$, $\tan\theta = 1/b\,(24ey - 2b^2)^{\frac{1}{2}}$,

where b = breadth EF, A = cross section ABDC, l = distance between centres of gravity and of buoyancy before displacement, y = depth of immersion AC, h = height EC, $\angle$ AOH = θ.

Hence $A = by$, $l = -\frac{1}{2}(h-y)$, $y = h\rho$; $\therefore A = bh\rho$, $l = -h/2\,(1-\rho)$;

therefore $$S = \left[\frac{b^2}{12h\rho} - h/2\,(1-\rho)\right]P\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

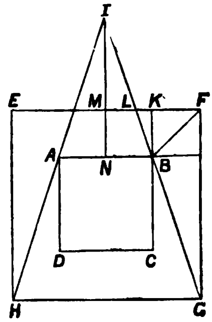
also $$\tan\theta = \left(\frac{12h^2(1-\rho)\rho}{b^2}\right)^{\frac{1}{2}} - 2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) we have, for stable equilibrium, $b/h > 6\rho(1-\rho)^{\frac{1}{2}}$; hence the ratio lies between the value stated in the question. Substituting $h^2 = \frac{10}{9}h^2$ in (2) and $\rho = 4/5$, we have

$$\tan\theta = (\tfrac{2}{15})^{\frac{1}{2}} = \cdot 365148,\quad \theta = 20^\circ\ 3'\ 35''.$$

4241. (Professor GENESE, M.A.)—A building a feet square has a walk b feet wide round it. Two persons are on the walk; find the chance that they can see each other.

Solution by Professors ZERR, NILKANTHA SARKAR, *and others.*



Let ABCD be the building, FGHE the outside of the walk. It is not possible that the walk is so wide that HA and GB will intersect within EFGH. There are three cases: (1) when in MLBN two corners of the walk and two of the building can be seen; and (2), (3), when in LBK, KBF three corners of the walk and two or three of the building can be seen.

Now $MNLB = \frac{1}{2}b\,(\frac{1}{2}a + x) = \frac{1}{2}b\left(a - \frac{a+b}{b^2}\right)$,

$LBK = \frac{1}{2}b\,(\frac{1}{2}a - x)$ $KBF = \frac{1}{2}b^2$,

NB, NL being axes of coordinates; hence

$$p = \text{chance} = \frac{1}{2b^2(a+b)^2}$$

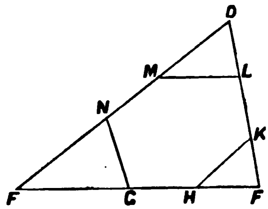
$$\times\left[\int_0^b\left\{\int_0^{\frac{1}{2}a}\tfrac{1}{2}b\,(\tfrac{1}{2}a + x)\,dx + \int_{\frac{1}{2}a-(a+b)/b^2}^{\frac{1}{2}a}\tfrac{1}{2}b\,(\tfrac{1}{2}a - x)\,dx + \int_{\frac{1}{2}a}^{\frac{1}{2}a+b}\tfrac{1}{2}b^2dx\right\}dy\right]$$

$$= \frac{3a^2b^4 + 4\,(a+b)^2 + 8b^6}{32b^4\,(a+b)^2}.$$

10272. (R. H. W. WHAPHAM, B.A.)—If straight lines be drawn parallel to the sides of the pedal triangle of a triangle ABC, so as to form with the intercepted portions of the sides of the pedal triangle an equilateral hexagon, prove that, if R be the circum-radius of ABC, the length of each side of the hexagon will be

$$R(\sin 2A \sin 2B \sin 2C) / (\sin 2B \sin 2C + \sin 2C \sin 2A + \sin 2A \sin 2B).$$

Solution by the PROPOSER, Professor CHAKRIVARTI, *and others.*



Let DEF be the pedal triangle of ABC; d, e, f its sides; and let GHKLMN be the equilateral hexagon; l the length of a side.

Then $EG = l \sin D/\sin E = ld/e$,

$\sin HF = ld/f$;

therefore $ld/e + ld/f + l = d$,

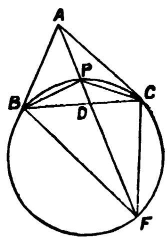
therefore $l = def/(ef + fd + de)$.

But $d = R \sin 2A$, $e = R \sin 2B$, $f = R \sin 2C$;

hence required result.

10704. (R. CHARTRES.)—Find a point P within a triangle ABC, such that the tangents from A, B, C respectively to the circumcircles of PBC, PCA, PAB shall be all equal.

Solution by the PROPOSER, Prof. RAMASWAMI AIYAR, *and others.*



Let P be the centroid of ABC; then, by similar triangles and Ptolemy's theorem,

$$PC : PB : PA = BF : CF : BC,$$

$$PC \cdot BF + PB \cdot CF + PA \cdot BC = AF \cdot BC;$$

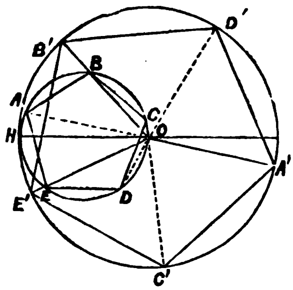
therefore $PC^2 + PB^2 + PA^2 = AF \cdot PA$ = square on tangent from A, and similarly for the other tangents; hence P is the point required.

Also $PA^2 + PB^2 + PC^2$ = a minimum.

10817. (E. M. LANGLEY, M.A.)—Show, by a geometrical proof, that applies to any regular $(2n+1)$-gon, that, if ABCDE is a regular pentagon, and O any point on the minor arc CD of its circumcircle, then $OA + OC + OD = OB + OE$. (L'HOSPITAL's *Sections Coniques*.)

Solution by Prof. Matz; E. M. Langley, M.A.; *and others.*

Let ABCDE be a regular pentagon, and O any point on minor arc CD; then $OA+OC+OD = OB+OE$.



Let OH be the diameter through O, and let OA, OB, OC, OD, OE meet the circle with centre O and radius OH in A′, B′, C′, D′, E′. Then (1) A′, B′, C′, D′, E′ are vertices of a regular pentagon, and (2) OA, OB, OC, OD, OE = the projections on OH of OA′, OB′, OC′, OD′, OE. But O is centroid of A′, B′, C′, D′, E′; therefore

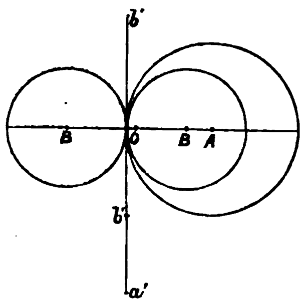
$$OA+OC+OD = OB+OE.$$

This investigation applies to any regular $(2n+1)$-gon.

10403. (J. C. St. Clair.) — Two unequal circles roll with equal angular velocity on a fixed straight line. Show that in every case the envelope of their radical axis is a parabola.

Solution by the Proposer.

We shall obtain the *form* of the envelope in every case by assuming for origin a point of common contact of the circles with the fixed line L. Let this line be the axis of y, and let the radius a be $> b$.



Then, for any other simultaneous points of contact a', b', we have

$$Oa' = ma, \quad Ob' = mb,$$

and the equations to the circles are

$$(y \pm ma)^2 + (x \pm a)^2 = a^2 \ldots\ldots (A),$$
$$(y \pm mb)^2 + (x \pm b)^2 = b^2 \ldots\ldots (B).$$

Subtracting, we get for the equation of the radical axis

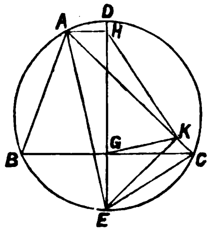
$$m^2(a^2-b^2) \pm 2my\,(a \pm b) \pm 2x\,(a \pm b) = 0,$$

the envelope of which is (Salmon's *Conics*, Art. 283), $2y^2 = \pm Cx$, a parabola having L for tangent at vertex.

[There are four pairs of equal and opposite parabolas with same or opposite sides and directions, $C = a \pm b$; opposite sides and same direction, $C = \dfrac{(a+b)^2}{a-b}$; same sides and opposite directions $C = \dfrac{(a-b)^2}{a+b}$; the sign of x being always contrary to that of the radius OA.]

10736. (R. H. W. WHAPHAM, B.A.)—ABC is a triangle inscribed in a circle; DE is a diameter bisecting the base BC at G; from E is drawn a perpendicular EK to one of the sides; and the perpendicular from the vertex on DE meets DE in H. Show that EK touches the circle GHK.

Solution by H. J. WOODALL; R. KNOWLES, B.A.; *and others.*



$\angle$ EGC = EKC = right angle;

therefore EGKC is cyclic; therefore

GKE = GCE = BCE = BAE = EAC,

because E is the mid-point of the arc BC; therefore EA bisects BAC.

$\angle$ AHE = AKE;

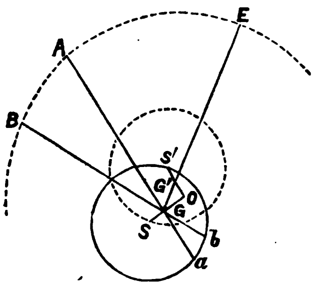
therefore AHKE is cyclic; therefore

$\angle$ EAK = EHK,

i.e., $\angle$ EHK = GKE, and EK is tangential to circle GHK.

10127. (J. C. ST. CLAIR.)—If n points be taken on a circle, prove that (1) the mean centres of the n systems of $n-1$ points, formed by omitting each point in succession, lie on a circle S_n; (2) if another point be taken on the original circle, the centres of the $n+1$ circles S_n, obtained by omitting each point in succession, lie on an equal circle, and so on *ad infinitum;* and (3) hence deduce a proof of Quest. 9997.

Solution by the PROPOSER.



(1) Let G be the mean centre of the system of n points A. Produce AG to a, making

$$AG : Ga = n-1 : 1.$$

Then a is the centre of the system of $n-1$ points formed by excluding A. In the same manner, we get

$$Gb = GB/n-1, \text{ \&c.};$$

and therefore the points a, b, c, d, &c. lie on a circle whose centre S lies on OG, and whose radius $= OA/n-1$. Also,

$$GO = (n-1)\,GS.$$

(2) Let E be any $(n+1)$th point on the original circle. Join EG and take on it a point G′, so that $G'E = n \cdot G'G$. Then G′ is the mean centre of the $n+1$ points A ... E. And, as above, the $n+1$ mean centres G lie on a circle whose centre lies on OG′, and whose radius $= R/n$. And, since $OS = \dfrac{n}{n-1}\,OG$, S lies on another circle whose centre S′ is on OG′,

and whose radius $= R/(n-1)$. Also,

$$OS' : OG' = R/(n-1) : R/(n+1) = n+1 : n-1;$$

and radius of circle S″ will be $R/(n-1)$, &c.

(3) If we put $n = 3$, the first circle S becomes the nine-points circle of the triangle ABC, and if a fourth point D be taken on the circle O, the four circles S pass through S′, which corresponds to the point Q in Question 9997.

9485. (Prof. Orchard, M.A., B.Sc.)—Solve, by a simple quadratic method, the equation $x^8-12x^6-10x^5+23x^4+50x^3+52x^2-40x-64 = 0$.

Solution by W. J. Greenstreet, M.A.; Prof. Chakrivarti; *and others.*

The equation reduces to

$$(x^2-1)(x^2-4)(x^2+x+2)(x^2-x-8) = 0,$$

giving the roots $\pm 1,\ \pm 2,\ \dfrac{-1\pm(-7)^{\frac12}}{2},\ \dfrac{1\pm(33)^{\frac12}}{2}.$

10547. (R. Knowles, B.A.)—Two conics intersect in ABCD; prove that the poles, with respect to each conic of (1) AB, CD; (2) AD, BC; (3) AC, BD are collinear.

Solution by Professors Anderson, Zerr, *and others.*

The polar of the point (α, β) with respect to the conic

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c = 0$$

is the line $\quad x(a\alpha+h\beta+g)+y(h\alpha+b\beta+f)+g\alpha+f\beta+c = 0,$

which, if (α, β) be the intersection of the lines $L = 0$, $M = 0$, is also its polar with respect to any one of the system of conics $S+\kappa LM = 0$. The poles of AB, CD with respect to the two conics in the Question, lie on the polars of their point of intersection; and since by the above, these polars coincide, they lie on the same straight line. The same proof applies to other cases.

9613. (Professor Hanumanta Rau, M.A.)—The intersections of the sides as well as the diagonals of a regular pentagon give the angular points of regular figures. If the sides and areas of these figures be represented respectively by a, b, c and A, B, C; prove (1) $b+c = 3a$, (2) $B+C = 7A$, (3) $(C-B)/(c-b) = 3A/a$.

Solution by W. J. Greenstreet, M.A.; Prof. Matz, M.A.; *and others*.

Trigonometry gives $b = \dfrac{2a}{3+\sqrt{5}}, \quad c = \dfrac{3+\sqrt{5}}{2}a, \quad \therefore\ b+c = 3a \quad\text{...(1)}.$

Again, figures are similar; therefore $\dfrac{B+C}{A} = \dfrac{b^2+c^2}{a^2} = 7$ ....(2),

and from (1), $b^2-c^2 = 3a(b-c) = \dfrac{3a^2(b-c)}{a}$ or $\dfrac{c^2-b^2}{3a^2} = \dfrac{c-b}{a}$;

therefore $$\frac{C-B}{3A} = \frac{c-b}{a} \quad\text{..............................(3).}$$

7139. (A. McMurchy, B.A.)—Prove that

$$\frac{1}{(\log x)^2} = \left(\frac{x^{\frac{1}{2}}}{x-1}\right)^2 + 2\left\{\left(\tfrac{1}{2}\cdot\frac{x^{\frac{1}{4}}}{x^{\frac{1}{2}}+1}\right)^2 + \dots \textit{ad inf.}\right\}$$

Solution by Professors Mukhopadhyay, Sarkar, *and others*.

$$\log x = (x-1)\,\frac{2}{x^{\frac{1}{2}}+1}\cdot\frac{2}{x^{\frac{1}{4}}+1}\cdot\frac{2}{x^{\frac{1}{8}}+1}\dots,$$

Differentiating, $\dfrac{1}{\log x} = \dfrac{x}{x-1} - \tfrac{1}{2}\left\{\dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}+1} + \tfrac{1}{2}\cdot\dfrac{x^{\frac{1}{4}}}{x^{\frac{1}{4}}+1} + \dots\right\};$

therefore $$\frac{1}{\log x} = \tfrac{1}{2}\cdot\frac{x+1}{x-1} - \tfrac{1}{2}\cdot\frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1} - \tfrac{1}{4}\cdot\frac{x^{\frac{1}{4}}-1}{x^{\frac{1}{4}}+1} - \dots,$$

by subtracting and adding $\tfrac{1}{2}$ or $\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\dots$ to sinister. Differentiating

again, $$\frac{1}{x(\log x)^2} = \frac{1}{(x-1)^2} + (\tfrac{1}{2})^2\cdot\frac{2x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}+1)^2} + (\tfrac{1}{4})^2\cdot\frac{2x}{(x^{\frac{1}{4}}+1)^2} + \dots,$$

whence we obtain the stated result.

10483. (Professor Steiner.)—Étant données trois circonférences O, O_1, O_2, passant par un même point A, mener par A une sécante, qui rencontre les trois courbes en trois B, B_1, B_2, tels que $BB_1 : B_1B_2 = m : n$.

Solution by Professors Anderson, Nilkantha Sarkar, *and others*.

The length AB is the projection of the diameter AD of the circle O on the line AB, and AB_1, AB_2 are the projections on the same line of the diameters AD_1, AD_2 of the other circles.

Regarding the lines AD, AD_1, AD_2 as representing forces, we have,

$$m\,(\text{component of } AD_2 - \text{component of } AD_1)$$
$$= n\,(\text{component of } AD_1 - \text{component of } AD).$$

Hence the following construction will determine the line AB:—Produce DA to D′, and D_1A to D_1', making $AD' = DA$ and $AD'_1 = D_1A$. Find the centre of mean position G of the points D′, D'_1, D_1, D_2, for the system of multiples n, n, m, m, and join AG. The perpendicular through A to AG is the required line.

9607. (Sarah Marks, B.Sc.)—Given

$$(x^2+y^2+z^2+c^2-a^2)^2 = 4c^2(x^2+y^2),$$

find the points the normals at which make angles α, β, γ with the axes, and the loci of points for which (1) γ is constant, (2) α is equal to β.

Solution by Emily Perrin; A. Gordon; *and others.*

Let $\quad \phi\,(xyz) \equiv (x^2+y^2+z^2+c^2-a^2)^2-4c^2(x^2+y^2) = 0.$

In polar coordinates this is $r^2+c^2-a^2 \pm 2cr\sin\theta$, and represents the surface generated by a circle of radius a, whose centre moves round a fixed circle of radius c, and whose plane contains the normal to the plane of the fixed circle through its centre. If $a < c$, this is an anchor-ring.

$$\phi(x) = 4x(x^2+y^2+z^2+c^2-a^2)-8c^2x, \quad \phi(z) = 4z(x^2+y^2+z^2+c^2-a^2);$$

therefore $\quad [\phi(x)]^2+[\phi(y)]^2+[\phi(z)]^2 = 16a^2(r^2+c^2-a^2)^2$

by the equation to the surface; therefore required points are given by

$$(r^2+c^2-a^2)(x \pm a\cos\alpha) = 2c^2x, \quad (r^2+c^2-a^2)(y \pm a\cos\rho) = 2c^2y,$$
$$(r^2+c^2-a^2)(z \pm a\cos\gamma) = 0.$$

(i.) If $a < c$, the points are $z = \pm a\cos\gamma$, $y = \pm x\cos\beta\sec\alpha$; (ii.) if $a > c$, we have also the points $r^2+c^2-a^2 = 0$, $x = y = 0$, *i.e.*, the two points $x = y = 0$, $z = \pm\sqrt{a^2-c^2}$.

1. If γ = constant, the locus consists of two circles

$$(x^2+y^2+c^2-a^2\sin^2\gamma)^2 = 4c^2(x^2+y^2), \quad z = \pm a\cos\gamma;$$

2. If $\alpha = \beta$, the locus consists of the ellipses

$$y = \pm x, \quad 2x^2+z^2+c^2-a^2 = \pm 2\sqrt{2}\,cx.$$

10022. (R. Sorreau.)—Si un nombre entier a, terminé par 1 ou par 5, est multiple de 3, plus 1, l'expression $(a-1)(a^2-a)(a^3-4a)$ est divisible par 43200.

Solution by J. J. BARNIVILLE; Professor EMMERICH*; *and others.*

$$a-1 = 5p = 3q = 15n; \text{ therefore } f(n) = (a-1)(a^2-a)(a^3-4a)$$
$$= 225n^2(225n^2+30n+1)(225n^2+30n-3)$$
$$= 675n^2(225n^2+30n+1)(75n^2+10n-1);$$
$$f(n+1) = 675(n+1)^2(225n^2+480n+256)(75n^2+160n+84).$$

By multiplication, $f(n+1)-f(n)$ is a multiple of 43200; therefore, if $f(n)$ is divisible, $f(n+1)$ is divisible; but we know it to hold when $n=1$, therefore it holds for every value of n.

10805. (Professor ZERR.)—Solve the equation

$$x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}$$
$$= x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1.$$

Solution by W. J. GREENSTREET, M.A.; R. KNOWLES, B.A.; *and others.*

The equation reduces to $(x^{15}-1)(x^{10}-1) = 0$;

therefore $\quad x = \cos\frac{2r\pi}{15}+\iota\sin\frac{2r\pi}{15}, \text{ or } \cos\frac{2r\pi}{10}+\iota\sin\frac{2r\pi}{10}.$

The values include one value (unity) of x, which must be rejected, as we have introduced a factor $x-1$.

10426. (M. MOUREAU.)—Sur une droite OA de longueur $2d$, on prend des points qui la partagent en $2n$ parties égales. Aux points de division, on applique des forces parallèles, mesurées par les distances de leurs points d'application au point O. Ces forces ont même direction, mais sont alternativement dirigées dans un sens et dans l'autre. Trouver (1) l'intensité de la résultante, et (2) la distance de son point d'application au point O.

Solution by Professors ANDERSON, ZERR, *and others.*

There are n forces in the direction in which the first force is drawn, and $n-1$ in the opposite direction. If the force at any point be $\mu\times$ distance from the end of O, the resultant is easily found by summation to be μd. And if the distance of the centre of parallel forces from O be (x),

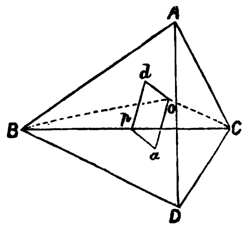
$$\mu d\,(x) = \mu d^2/n^2\left[1^2-2^2+3^2-4^2+\ldots-(2n-2)^2+(2n-1)^2\right],$$

or $\quad (x) = d/n^2(1+5+9+13+\ldots+4n-3) = 2d-d/n.$

The resultant coincides with the last force and is equal in magnitude to that at the middle point of the rod.

10655. (Professor WOLSTENHOLME, M.A., Sc.D.)—In a tetrahedron ABCD, the sums of the lengths of two pairs of opposite edges are equal (AB + CD = AC + BD): prove that the sums of the corresponding pairs of dihedral angles are also equal [(AB) + (CD) = (AC) + (BD)].

Solution by Professors RAMASWAMI AIYAR, BEYENS, *and others.*



Since BA − AC = BD − DC, the perpendiculars from the incentres a, d of the faces A, D on the line BC meet it in the same point p; hence the perpendiculars at a, d to the faces A, D meet in a point O. And it is easy to see that the lines BO, CO make equal angles with the edges of the solid angles B, C. Thus the difference of the dihedral angles (AB), (BD) = the difference of the angles into which the plane BOC divides the dihedral angle (BC) = the difference of the dihedral angles (CA), (CD); therefore (AB) + (CD) = (AC) + (BD).

10672. (W. J. GREENSTREET, M.A.)—ABC is a triangle. From any point D within the triangle, perpendiculars DM and DN are dropped on AB, AC. If CN . AC = BM . AB, find the locus of D.

Solution by Professors ANDERSON; R. KNOWLES, B.A.; *and others.*

The four points A, M, D, N lie on a circle of which AD is a diameter, and of which O, the mid-point of AD, is the centre.

Since CN . CA = BM . BA, the tangent from C to this circle is equal to the tangent from B. Hence CO = BO, and the locus of O is a straight line bisecting BC at right angles. The locus of D is therefore a straight line, parallel to the locus of O, and whose distance from A is twice the distance of that line from the same point.

6942. (By E. RUTTER.)—Through the focus F of a parabola draw a circle, having its centre in the principal axis, and cutting the axis in A and the directrix in B, C, such that the triangle ABC equals the triangle TPT′, formed by two tangents PT, PT′, and their chord of contact TT′.

Solution by R. KNOWLES, B.A.; W. H. BLYTHE, B.A.; *and others.*

The area of the polar triangle (TPT′) of a parabola is $\dfrac{(k^2-4ah)^{\frac{3}{2}}}{2a}$,

(hk) being the pole of TT′; and if equation to circle ABC be $x^2+y^2+dx+ey+f=0$, we find (1), from its passing through the focus, $f=-(a^2+ad)$; (2) its centre being in the axis $e=0$; (3) it meets the axis again in A abscissa $=-(a+d)$, therefore perpendicular from A on directrix $=d$; (4) $x=-a$ gives $y^2{}_2=xad$, the points where it meets the directrix; therefore $\frac{k^2-4ah}{2a}+y^2=2ad$, and the equation to the circle is found in terms of the pole of TT′

$$x^2+y^2+\frac{k^2-4ah}{2a}x-\frac{2a^2+k^2-4ah}{2}=0.$$

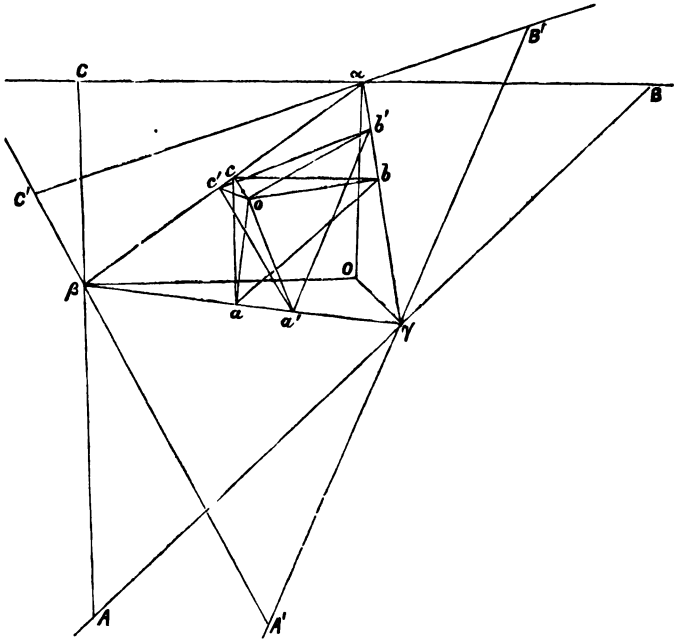
[It is suggested that the triangle or circle TPT′ should, for shortness' sake, be denominated the polar triangle or circle of TT′.]

7053. (Professor Wolstenholme, M.A.)—A triangle ABC of given form (*i.e.*, whose angles are given) is circumscribed to a given triangle $\alpha\beta\gamma$, and another triangle abc is inscribed in the same given triangle so as to have its sides parallel to the corresponding sides of the triangle ABC. When ABC is a maximum, of course the normals to its sides at α, β, γ meet in a point (O); abc will then be a minimum, and the normals at a, b, c to the sides of the given triangle meet in a point (o) such that O, o are foci of a conic inscribed in $\alpha\beta\gamma$. In any other positions of ABC, abc, the points O, o still divide the corresponding triangles in the same ratios, and if θ be the angle through which the sides have turned from the maximum or minimum position, the sides of ABC will be diminished and those of abc increased in the ratio $\cos\theta : 1$ (the area of $\alpha\beta\gamma$ being always a mean proportional between those of abc, ABC). We will call any point P, whose areal coordinates referred to ABC are fixed, a point belonging to ABC; and similarly for any point p belonging to abc. So also we may have straight lines or curved lines *belonging* to either triangle, moving with it, and each diagram being similar in any position to the corresponding diagram when the triangles are respectively maximum and minimum. Then (1) the locus of any point P belonging to ABC is a circle whose diameter is OP_1, where P_1 is the initial position of P, and the locus of any point p belonging to abc is a straight line through p_1; (2) any straight line belonging to abc passes through a fixed point, the foot of the perpendicular upon it from O in its initial position, and the envelope of any straight line belonging to abc is a parabola which initially touches at the vertex; (3) the envelope of any circle belonging to ABC is a limaçon of which O is a focus, and axis the diameter of the circle through O in its initial position, and the envelope of any circle belonging to abc is a conic of which one focus is o, and centre is the initial centre of the circle which is initially the auxiliary circle of the conic; (4) the envelope of any curve U belonging to ABC is the pedal with respect to O of U_1 (the initial position of U), and of any curve u belonging to abc is the negative pedal with respect to o of u_1. Of straight lines and points in the figure not *belonging* to either of the triangles, OA, OB, OC meet the sides of the triangle $\alpha\beta\gamma$ in points A′, B′, C′, such that B′C′, C′A′, A′B′ each envelop a conic having a focus at O and touching two sides of $\alpha\beta\gamma$, which three conics have one

real common tangent; and that Aa, Bb, Cc meet in a point whose locus is an hyperbola, having O for the vertex and axis along Oo.

Solution by the PROPOSER.

Let $\alpha\beta\gamma$ be a fixed triangle, $a'b'c'$ a triangle inscribed in $\alpha\beta\gamma$, of given form, *i.e.*, having its angles equal to given angles A, B, C, and A′B′C′ another triangle circumscribing $\alpha\beta\gamma$ and having its sides parallel to those of $a'b'c'$, and consequently also having angles A, B, C. Then the circles $b'c'\alpha$, $c'a'\beta$, $a'b'\gamma$ concur in one point o; from o draw oa, ob, oc perpendicular to the sides of $\alpha\beta\gamma$, and join a, b, c, forming the triangle abc. Then,



because a circle goes round $obac$, the angle $boc = \pi - \alpha$, and, for a similar reason, the angle $b'oc'$ also $= \pi - \alpha$; hence the angles boc, $b'oc'$ are equal, whence also the angles bob', coc' are equal, and each is equal to aoa'. Call each of these angles θ. Then

$$\frac{oa'}{oa} = \frac{ob'}{ob} = \frac{oc'}{oc} = \sec\theta,$$

and the angles boc, coa, aob being equal respectively to $b'oc'$, $c'oa'$, $a'ob'$, the diagrams $oa'b'c$, $oabc$ are similar, o being their centre of similitude;

and every length belonging to the former will bear to the corresponding length belonging to the latter the ratio $\sec\theta : 1$, and will make with it an angle θ. We shall consider the diagram $a'b'c'$ to be revolving about o, always preserving its shape and altering its magnitude according to the law investigated; and, denoting any point fixed with respect to $a'b'c'$ by any letter, as p', we shall denote the corresponding point in abc by p, and call p the initial position of p'. Similarly with groups of points, or figures of any kind. The angles $\beta o\gamma$, $\gamma o\alpha$, $\alpha o\beta$ are respectively $A+\alpha$, $B+\beta$, $C+\gamma$. Draw a triangle ABC circumscribing $\alpha\beta\gamma$, and having its sides parallel to those of abc; then the circles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ concur in a point O, at which the sides $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ subtend angles $\pi-A$, $\pi-B$, $\pi-C$; also the circles $A'\beta\gamma$, $B'\gamma\alpha$, $C'\alpha\beta$ concur in the same point O. The perpendiculars from O on the sides of the triangle $A'B'C'$ will be equal to $O\alpha\cos\theta$, $O\beta\cos\theta$, $O\gamma\cos\theta$; hence the diagram $OA'B'C'$ is similar to the diagram OABC, every length being diminished in the ratio $\cos\theta : 1$, and turned through an angle θ. We shall consider $A'B'C'$ as turning about the point O at the same rate as $a'b'c'$ about o, preserving its shape and altering its size, and consider ABC as the initial position of $A'B'C'$, and use a notation similar to the one already described. It is obvious that $a'b'c' = abc\,.\sec^2\theta$, and $A'B'C' = ABC\cos^2\theta$, so that the product of the two areas is constant; and it is easily proved that the triangle $\alpha\beta\gamma$ is the mean proportional to abc, ABC, and therefore to $a'b'c'$, $A'B'C'$. Also the triangle $A'B'C'$ is a maximum, and the triangle abc a minimum, in their respective initial positions. (Of course this would follow from the normals at α, β, γ in the one case, and at a, b, c in the other, then concurring each in one point.) Since the angles $\beta o\gamma$, $\beta O\gamma$ are together $= \pi+\alpha$, and similarly for the others, we see that o, O are foci of a conic inscribed in α, β, γ. Now consider what will be the locus of any point p' belonging to $a'b'c'$. We have $op' = op\sec\theta$, and the angle $pop' = \theta$. Hence the locus of p' is the straight line through p at right angles to op—*i.e.*, the locus of any point belonging to $a'b'c'$ is a straight line. Secondly, for the envelope of any straight line, let ok' be the perpendicular on it; then k' lies on a fixed straight line (by the property just proved), and the envelope is a parabola whose focus is o and vertex k. Thirdly, to find the envelope of any circle belonging to $a'b'c'$: let p' be its centre, take op for axis of x, and origin at p. Then, if $op=c$, and a be the radius of the circle in its initial position, the equation will be

$$x^2+(y-c\tan\theta)^2 = a^2\sec^2\theta,$$

or

$$(a^2-c^2)\tan^2\theta+2cy\tan\theta+a^2-x^2-y^2 = 0;$$

and that of the envelope is

$$c^2y^2 = (a^2-c^2)(a^2-x^2-y^2), \quad \text{or} \quad \frac{x^2}{a}+\frac{y^2}{a^2-c^2} = 1,$$

a conic whose centre is p, focus o, and to which the initial position of the circle is auxiliary circle. It is obvious that this conic is an ellipse or hyperbola according as o lies within or without the circle; and reduces to two points when o lies on the circle. We will take the corresponding questions as to $A'B'C'$ in the same order. First, for a point P', $OP' = OP\cos\theta$, and the angle $POP'=\theta$; hence the locus of P' is a circle passing through O, the initial position of P' being the other end of the

diameter through O. Secondly, for a straight line to which OK′ is perpendicular, OK′ = OK $\cos\theta$ and KOK′ = θ, hence the straight line always passes through K; or every straight line belonging to A′B′C′ passes through a fixed point (as B′C′, C′A′, A′B′ do by construction). Thirdly, for a circle whose centre is P′, take OP as axis, and O as origin, and let OP = c, and the radius of the circle in its initial position be a. The coordinates of the centre of the circle in its moving state will be $c\cos^2\theta$, $c\sin\theta\cos\theta$, and its radius $a\cos\theta$. Hence the equation is

$$(x-c\cos^2\theta)^2+(y-c\sin\theta\cos\theta)^2=a^2\cos^2\theta,$$

or $2x^2+2y^2-2cx\,(1+\cos 2\theta)-2cy\sin 2\theta+c^2(1+\cos 2\theta)=a^2(1+\cos 2\theta)$,

and the envelope is $(2x^2+2y^2-2cx+c^2-a^2)^2=(2cx-c^2+a^2)^2+4c^2y^2$,

or $$(2x^2+2y^2)\,(2x^2+2y^2-4cx+2c^2-2a^2)=4c^2y^2;$$

or, in polar coordinates (probably preferable from the first),

$$r^2-2cr\cos\theta+c^2-a^2=c^2\sin^2\theta,\quad i.e.,\quad r=c\cos\theta\pm a.$$

The two equations represent one single curve, since one can be formed from the other by writing $(-r\,.\,\pi+\theta)$ for (r,θ); and the envelope is a limaçon having a node (and therefore double focus) at O; this node being a crunode or acnode, according as O lies without or within the circle. The axis is the diameter of the circle in its initial position.

The triangles A′B′C′, $a'b'c'$ being in perspective, A′a', B′b', C′c' meet in a point whose locus we can see to be a conic by considering that it will be at infinity when the triangles A′B′C′, $a'b'c'$ are equal to each other and therefore to $\alpha\beta\gamma$. This gives the equation

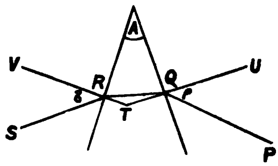
$$\cos^2\theta = \text{area of triangle } abc \div \text{area of triangle } \alpha\beta\gamma,$$

giving two values for θ, which will be real so long as the distance of o from the centre of the circle $\alpha\beta\gamma$ bears to the radius of that circle a ratio less than $1:\sqrt{3}$. For this particular value the conic locus must be a parabola, and for points at a greater distance the locus will be an ellipse. The point O is a point on the locus (when $\theta=\frac{1}{2}\pi$), and I expect that Oo is an axis, and that the other vertex corresponds to $\theta=0$, but I have not yet tried to verify these guesses. If A′O, B′O, C′O meet the sides of $\alpha\beta\gamma$ in points a'', b'', c'', the angle b''Oc'' = B′OC′ = A + α, or is constant, and hence $b''c''$, $c''a''$, $a''b''$ each envelops a conic with focus O and touching two sides of the triangle $\alpha\beta\gamma$. I inferred, I think by reciprocation on O, that these three conics have always one real common tangent. There is no doubt that many other interesting loci and envelopes may be discovered connected with these triangles, as well as many interesting special cases: *e.g.*, A = α, B = β, C = γ, when the envelope of the circle $a'b'c'$ is the conic inscribed in $\alpha\beta\gamma$ and having for its foci the centre of the circumscribed circle and centre of perpendiculars (o and O for this case). Also A = $\pi-2\alpha$, &c.; and A = $\frac{1}{2}(\pi-\alpha)$, &c., in which case o, O coincide. I may fairly leave these to my juniors to work out.

[Constructions for inscribing in a given triangle a triangle whose sides are parallel to those of another given triangle, and for inscribing or circumscribing triangles of given form and of given area, or of maximum or minimum area, are at once supplied by the above.]

10781. (J. J. WALKER, F.R.S.)—If A be the refracting angle of a prism, and θ, ϕ the differences between the angles which the part of the ray within the prism makes with the normals to the faces, and between the deviations at incidence and emergence respectively, while D is the total deviation; prove (1) that $\sin \frac{1}{2}(A+D) = m \cos\theta \sin \frac{1}{2}A / \cos(\theta+\phi)$, and thence (2) deduce the position of minimum deviation.

Solution by Professors ZERR, PROMANATHA DATA, *and others.*



(1) Let PQRS be the ray, $\angle PQU = \rho$, $\angle RQT = \rho'$, $\angle VRS = \delta$, $\angle QRT = \delta'$; then from the equations $\sin\rho = m\sin\rho'$, $\sin\delta = m \sin\delta'$, remembering that

$$\rho'+\delta' = A, \quad \text{and} \quad \rho+\delta = A+D,$$

we get $\quad \sin\frac{1}{2}(A+D)$

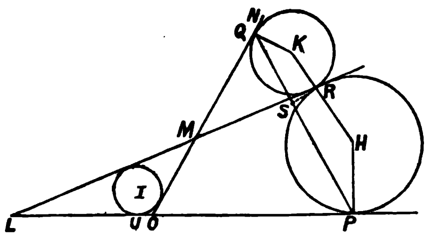
$$= m \sin\tfrac{1}{2}A \cos\tfrac{1}{2}(\rho'-\delta') \sec\tfrac{1}{2}(\rho-\delta);$$

whence, writing $\frac{1}{2}(\rho'-\delta') = \theta$ and $\frac{1}{2}\{(\rho-\rho')-(\delta-\delta')\} = \phi$, or $\frac{1}{2}(\rho-\delta) = \theta+\phi$, we have the result stated.

(2) $\theta = \phi = 0$ gives the position of minimum deviation, because if $\theta = 0$, and therefore $\phi = \theta$, $\cos\theta/\cos(\theta+\phi) = 0$; but if θ has a small value, so has ϕ, and consequently $\cos\theta/\cos(\theta+\phi) > 1$.

10845. (Professor BOYS, F.R.S.)—Given two lines OA, OB, intersecting at any angle, also points P, Q, one in each line; find (1) two circular arcs, PR, QR, with radii as $m : n$, to touch the given lines at P and Q, and each other at R; and (2) give also a construction.

Solution by Professor GENESE; H. FORTEY, M.A.; *and others.*



Let any two circles touch the lines at P, Q, and each other at R; then the locus of R is a circle. For, let the common tangent at R meet OP, OQ at L, M;

$$LM = LR - MR = LP - MQ = OP + LO - OQ + OM.$$

Therefore $\quad LO + OM - LM = OQ - OP = b - a$, say.

Hence, if the incircle of LOM touch OL at U, $OU = (b-a)/2$ is known, as is the angle LOM; thus the incircle is fixed: let its centre be I. Then $IR = IP$, and locus of R is a circle, with centre I.

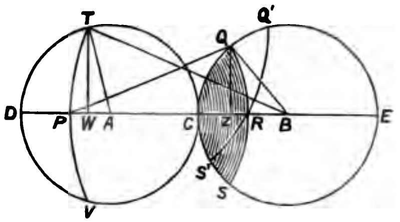
Again, let H, K be the centres of the required circles; then

$$HR : RK :: HP : KQ :: m : n$$

Divide PQ at S, so that PS : SQ :: $m : n$. By Bellavitis's Theorem, SR is equally inclined to QK, PH, and is thus a known straight line whose intersections with the above locus of R give solutions of the problem. (If P, Q be particles simultaneously describing PH, QK with *equal* momenta, SR is the locus of their mass-centre.)

10802. (The late Professor Seitz.)—Two equal spheres touch each other externally. If a point be taken at random within each sphere, show that (1) the chance that the distance between the points is less than the diameter of either sphere is $\frac{13}{35}$, and (2) the average distance between them is $\frac{11}{5}r$.

Solution by D. Biddle.



1. Let A, B, C be respectively the centres and point of contact of the two spheres, and let CTDV, CQES be great circles lying in the same plane. Take any point P in CD, and with radius = CD describe the arc QRS. Then, if QCSRQ be supposed to rotate about the axis DE, it will generate that section of the sphere (B), within which the second point must be taken, to fulfil the conditions when P is the first point.

Again, from B as centre, with radius BP, describe the arc TPV. Then, if TPV be supposed to rotate about the axis DE, it will generate a spherical superficies or film of points, all requiring the second random point to fall within equal sections of the sphere (B).

The area of this film is given by $2\pi.BP.PW$ (W being the foot of the perpendicular from T). Let the radius of each sphere = unity, and $AP = \pm x$. Then $BP = 2-x$, and $AW = PW+x$. Also, we have

$$TW = \{1-(PW+x)^2\}^{\frac{1}{2}} = \{(2-x)^2-(2-PW-x)^2\}^{\frac{1}{2}},$$

whence $PW = \frac{1}{4}(1-x^2)$. Therefore the area of the film $= \frac{1}{2}\pi(2-x)(1-x^2)$, and its ratio to the entire sphere ($= \frac{4}{3}\pi$) may be represented by

$$\tfrac{3}{8}(2-x)(1-x^2)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a).$$

Now, the double convex lens-like portion of B may be regarded as consisting of a series of concentric films, as represented in the figure, each having P as its centre; and if y = the variable radius, the area of any such film $= \pi y\{1-(2-x-y)^2\}/(2-x)$.

At present, the range of y is $(1-x)$ to 2. Consequently, the solid contents of the lens and its ratio to the entire sphere are given by

$$\pi\int_{1-x}^{2}\frac{\{1-(2-x)^2\}\,y+2\,(2-x)\,y^2-y^3}{2-x}\,dy$$
$$=\pi\left(\tfrac{1}{12}x^4-\tfrac{2}{3}x^3-\tfrac{1}{2}x^2+\tfrac{4}{3}x+\tfrac{13}{12}\right)/(2-x),$$
$$\tfrac{1}{16}\,(x^4-8x^3-6x^2+16x+13)\,/\,(2-x)\ \ldots\ldots\ldots\ldots\ldots\ldots\ (\beta).$$

Multiplying (β) by (α), and integrating, we have

$$\frac{3}{128}\int_{-1}^{+1}(13+16x-19x^2-24x^3+7x^4+8x^5-x^6)\,dx$$
$$=\frac{3}{128}\left(26+0-\frac{38}{3}-0+\frac{14}{5}+0-\frac{2}{7}\right)=\frac{3\,(1664)}{128\,(105)}=\frac{13}{35}.$$

2. In this case the range of y is $(1-x)$ to $(3-x)$. Each film in (B), as before, has an area $=\pi y\,\{1-(2-x-y)^2\}/(2-x)$. Consequently, the average distance of all points in (B) from P in (A), found by multiplying the area of each film by ry, integrating, and dividing by $\frac{4}{3}\pi$ (= sphere), is given by

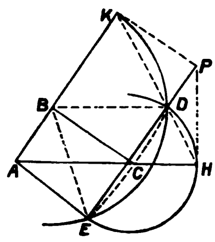
$$\tfrac{3}{4}r\int_{1-x}^{3-x}\frac{\{1-(2-x)^2\}\,y^2+2\,(2-x)\,y^3-y^4}{2-x}\,dy=\tfrac{3}{4}r\left(\tfrac{4}{3}x^2-\tfrac{16}{3}x+\tfrac{28}{5}\right)/(2-x)$$
$$=\tfrac{1}{5}r\,(5x^2-20x+21)/(2-x)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\gamma).$$

Multiplying (γ) by (α), and integrating, we have

$$\tfrac{3}{40}r\int_{-1}^{+1}(21-20x-16x^2+20x^3-5x^4)\,dx=\tfrac{3}{40}r\left(42-0-\tfrac{32}{3}+0-2\right)=\tfrac{11}{5}r.$$

10850. (Editor.)—In a triangle ABC there are given the angle A and the sum of the sides AB, AC; and around B, C as centres, with AC, AB, as radii respectively, the circles KDE, DHE are drawn; find (1) the loci of D, E; and (2) prove that DE passes through a fixed point.

Solution by J. C. St. Clair; H. Fortey; *and others.*



Upon the sides AC, AB, take

$$AH = AK = AB + AC,$$

and draw the perpendiculars HP, KP; then the tangents PH, PK being evidently equal, P lies on DE, the radical axis of the circles; *i.e.*, DE passes through the fixed point P.

(1) Since ABDC is a parallelogram,

$$\angle DCH = KAH;$$

and the triangles DCH, KAH being isosceles, it follows that $\angle$ CHD = AHK; hence D lies on HK. Again, the triangles BEC, CAB have their sides equal, and the base BC common; hence AE is parallel to BC. But the line of centres BC is

perpendicular to the common chord DE; thus PEA is a right angle, and E lies on the circle AHPK.

[Professor SCHOUTE proves the theorem analytically, thus:—

Let A be the origin, AX the internal, and AY the external bisector of the angle $A = 2\alpha$. Let AB and AC be represented by $a+z$ and $a-z$. Then, for the coordinates of B, C, D (fourth vertex of parallelogram ABDC), we find $\{(a+z)\cos\alpha,\ (a+z)\sin\alpha\}$, $\{a-z)\cos\alpha,\ -(a-z)\sin\alpha\}$, $\{2a\cos\alpha,\ 2z\sin\alpha\}$, and for the equation of the line DE perpendicular to BC through D, $y-2z\sin\alpha+z\,(x-2a\cos\alpha)/a = 0$.

The last result proves that DE passes through a fixed point F on AX; the previous result shows that D describes a right line parallel to AY; and then it is clear that the vertex E of the right angle AEF describes the circle of which AF is a diameter].

7933. (Rev. T. P. KIRKMAN, M.A., F.R.S.)

Thirteen at the board! they gaily mock
At ancient fear and awe.
Then, gloom and thunder—a baleful shock
Unmans them all, and, conscience-struck,
They hovering o'er them saw,
Frowning in flame, the angry Puck,
Who cried: "D'ye brave the law?
Before another year can fly,
Some shall sicken and one shall die."

Small boot to tell what wail and moan
Arose; ye better far
Read how, by softened Puck, was shown
The way that woe to bar.

"Fail not for a year, when the moon is [round,
That four of you repair,
For her noon half-hour, to the rustling [mound
Of Druid rites; and there,
Pacing slow the sacred ring,
With drooping foreheads, softly sing,
In weather foul or fair,
Praise to the Fairy Queen and King;
Then loud, when noon is gone,—
'Titania loveliest, regal Oberon,
Command that Puck
Ward off ill-luck
From the sorrowing twelve and one.'

"New fours, for thirteen moons, be told
Their penitent watch on the hill to hold;
But no two twice, of the banned thirteen,
May see together the moonlit scene.'

Solution by H. J. WOODALL.

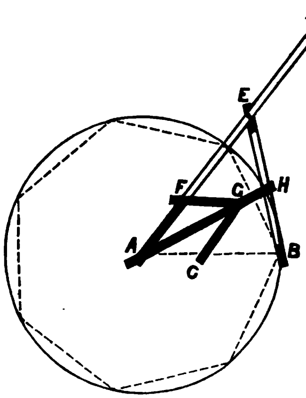
A	B	C	D
A	E	F	G
A	H	I	J
A	K	L	M
B	E	H	K
B	F	I	L
B	G	J	M
C	E	J	L
C	F	H	M
C	G	I	K
D	E	I	M
D	F	J	K
D	G	H	L

The arrangement of "THE BANNED THIRTEEN" is as set forth in the annexed scheme.

A, B, C, D together first.
A takes the rows of the matrix,
B takes the columns of the matrix,
C takes the negative diagonals of the matrix,
D takes the positive diagonals of the matrix,
(*i.e.*, negative from right to left, positive from left to right.)

10865. (Alfred A. Robb.)—**Show that, by the aid of Peaucellier-cells, a machine may be constructed which will solve the problem of the inscription of a regular heptagon in a circle, within the limits of Euclidean geometry.**

Solution by the Proposer.



Let AB be a straight line.

Take two bars, AH and BE, each equal in length to AB, and let them turn on pivots about A and B respectively.

In the bar BE let there be a slit in the direction of its length, and let a pin attached to H move along in it.

Take another bar AX about equal to twice AB in length, and let it also have a slit in the direction of its length in which moves a pin attached to E.

From AB cut off a part AC, which may be of any length, but is best, for the working of the instrument, to be about equal to $\frac{1}{2}$AB.

Take a point F in AX so that AF = AC, and take two equal bars CG and FG connected by a pivot at G and moving on pivots attached to C and F.

Turn bar AH round A till it passes through the point G.

Then $\qquad \angle AHB = AEH + HAE = BAE + HAE,$

but $\qquad \Delta ACG \equiv \Delta AFG$ by construction of instrument;

$\therefore\ \angle HAE = HAB; \quad \therefore\ \angle AHB = BAE + HAB = 3$ times HAB ;

but $\qquad \angle ABH = AHB$, because $AB = AH$;

therefore $\qquad HAB = \frac{1}{7}$ of two right angles;

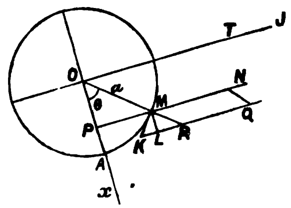
therefore $\angle BAE = \frac{1}{7}$ of four right angles = the angle subtended at the centre by one of the sides of a regular heptagon inscribed in a circle with centre A, and radius AB.

If the motion of the pins in the slits is objected to as not being within Euclid's limits of compasses and rulers, Peaucellier-cells may be attached to each of the bars BE and AX, so that they may describe their lines along each of these bars. If then the points H and E are attached by pivots to the points of the Peaucellier-cells which move along the bars, we have an instrument made entirely of compasses.

If, by a method similar to that by which it was bisected, the angle BAE was divided into 4, 8, 16, &c. equal parts; then, by turning AH into the proper position, we could form regular figures of $2^n - 1$ sides, where n is any positive integer.

10729. (Professor KERLHOFF.)—Étant donné un cercle O, on abaisse d'un point M de sa circonférence une perpendiculaire MP sur un rayon fixe OA, puis on prolonge PM de MN = n . OP. (1) Démontrer que le lieu de N est une ellipse; (2) trouver les axes de cette courbe; et (3) construire la tangente en N.

Solution by H. J. WOODALL; Professor ZERR; *and others.*



(1) $OP = x$, $PM = (a^2-x^2)^{\frac{1}{2}}$, $MN = nx$,

$\therefore \quad y = (a^2-x^2)^{\frac{1}{2}}+nx,$

$\therefore \quad a^2-x^2 = (y-nx)^2;$

i.e., $x^2(1+n^2)-2n \,.\, xy+y^2 = a^2,$

an ellipse, since $h_1^2-a_1b_1<0$.

(2) To find the axes, which obviously pass through O, turn the axes of x and y through an angle θ, where

$$\tan 2\theta = \frac{h_1}{a_1-b_1} = \frac{-2n}{n^2+1-1} = -\frac{2}{n}.$$

(3) To find the tangent at N. Draw tangent MK at M, draw ML parallel to axis of x, and KR parallel to axis of y; make $KR = n ML$, join MR, then draw NQ parallel to MR. NQ will be the required tangent at N.

$$[KQ = KR+RQ = n(ML+PO)].$$

8644. (The EDITOR.)—If a wire hoop be cut at random into three parts, prove that the respective probabilities (p_1, p_2) that the three pieces will, when straightened, admit of being formed into (1) a triangle of any kind, (2) an acute-angled triangle, are

$$p_1 = \tfrac{1}{4}, \quad p_2 = 3\log_e 2-2 = \tfrac{2}{25} \text{ nearly}.$$

Solution by D. BIDDLE.

(1). P_1 may be anywhere on the circumference (Fig. I.). Through O, the centre, draw P_1A; then P_2 may be anywhere between P_1 and A on either side. Through O draw P_2B; then it is clear that P_3 must lie between A and B. Let the circumference = unity, and P_1P_2 (arc) = x. Then we have the following integral: $2\int_0^{\frac{1}{2}} x \,.\, dx = \frac{1}{4}$, the required probability as to the formation of a triangle of *any* kind.

(2) Let P_1, P_2 (Fig. II.) be the foci of an ellipse in which $P_1P_2 = x$, and the sum of the focal distances of any point upon the perimeter = $1-x$. We are then able to determine the limits between which P_3 must lie in order that the triangle may be acute-angled. Upon P_1P_2 describe a

semicircle; this may (as in the figure), or may not, intersect the ellipse.

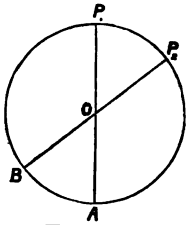


FIG. I.

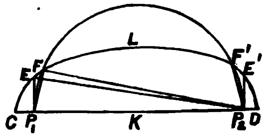


FIG. II.

The limits of x, for this intersection, are $\sqrt{2}-1$ and $\frac{1}{2}$; for non-intersection, 0 and $\sqrt{2}-1$. Moreover, the semi-latus rectum, P_1E, invariably indicates the minimum length of the second piece (y), and this we find from the following equation: $(1-x-y)^2-y^2-x^2=0$, whence $y=(1-2x)/(2-2x)$. Then, for x between the limits 0 and $\sqrt{2}-1$, the upper limit of y is $\frac{1}{2}(1-x)$; but, for x between $\sqrt{2}-1$ and $\frac{1}{2}$, it is represented by P_1F, and is found from the equation $(1-x-y)^2+y^2-x^2=0$, whence $y=\frac{1}{2}\left[1-x\pm\left\{(1+x)^2-2\right\}^{\frac{1}{2}}\right]$. And, as P_2E' to P_2F' are equally available, we have

$$4\left\{\int_0^{\sqrt{2}-1}\left(\frac{1-x}{2}-\frac{1-2x}{2-2x}\right)dx\right.$$
$$\left.+\int_{\sqrt{2}-1}^{\frac{1}{2}}\left(\tfrac{1}{2}\left[1-x-(x^2-2x-1)^{\frac{1}{2}}\right]-\frac{1-2x}{2-2x}\right)dx\right\}$$
$$=2\left\{\int_0^{\frac{1}{2}}\left(1-x-\frac{1-2x}{1-x}\right)dx-\int_{\sqrt{2}-1}^{\frac{1}{2}}(x^2+2x-1)^{\frac{1}{2}}\,dx\right\}$$
$$=2\left\{\int_0^{\frac{1}{2}}\frac{x^2\,.\,dx}{1-x}-\int_{\sqrt{2}-1}^{\frac{1}{2}}(x^2+2x-1)^{\frac{1}{2}}\,dx\right\}$$
$$=2\left\{(\log 2-\tfrac{5}{8})-(\tfrac{3}{8}-\log 4+\log 2\sqrt{2})\right\}$$
$$=3\log 2-2=\cdot 07944154,$$

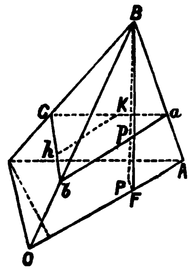
or rather more than $\frac{3}{38}$.

The probabilities (1) and (2) are precisely those which obtain in the case of a straight line divided at random into three portions.

10745. (S. TEBAY, B.A.)—a, b, c are conterminous edges of a tetrahedron; A_1, A_2, A_3 the areas of the faces contained by bc, ca, ab; A the area of the base, and x, y, z the radii of gyration about a, b, c; show that

$$(ax)^2+(by)^2+(cz)^2=A_1^2+A_2^2+A_3^2-\tfrac{1}{3}A^2.$$

Solution by the PROPOSER.



Let $OA = a$, $OB = b$, $OC = c$, and let the perpendicular BP cut the parallel section abc in p. Draw CE, PF perpendicular to OA, and let hk be parallel to ab, and at a distance v from c. Let $BP = p$, $Bp = u$, $CE = m$, $PF = n$, $BF = l$, λ, μ, ν, the cosines of the dihedral angles at OA, OB, OC. Then $hk = av/m$; and the square of the distance of hk from

$$OA = (p-u)^2 + 1/p^2 \{(m-n)\,u + p\,(n-v)\}^2;$$

therefore $$K^2M = \iint \left[(p-u)^2 + 1/p^2 \{(m-n)\,u + p\,(n-v)\}^2\right] du\, dv.$$

The integrals taken from $v = 0$, $v = mu/p$, and $u = 0$ to $u = p$ give

$$K^2M = \frac{map}{60}(m^2 + mn + l^2).$$

Since $M = \frac{1}{6}map$, therefore $K^2 = \frac{1}{10}(m^2 + mn + l^2)$; but

$$m = \frac{2A_2}{a},\quad n = \frac{2\lambda A_3}{a},\quad l = \frac{2A_3}{a};$$

therefore $$(ax)^2 = \tfrac{2}{5}(A_2^2 + A_3^2 + \lambda A_2 A_3).$$

So $(by)^2 = \frac{2}{5}(A_3^2 + A_1^2 + \mu A_3 A_1)$, and $(cz)^2 = \frac{2}{5}(A_1^2 + A_2^2 + \nu A_1 A_2)$.

$\therefore\ (ax)^2 + (by)^2 + (cz)^2 = \frac{2}{5}(2A_1^2 + 2A_2^2 + 2A_3^2 + \lambda A_2 A_3 + \mu A_3 A_1 + \nu A_1 A_2)$.

But $$A^2 = A_1^2 + A_2^2 + A_3^2 - 2\lambda A_2 A_3 - 2\mu A_3 A_1 - 2\nu A_1 A_2;$$

therefore $$(ax)^2 + (by)^2 + (cz)^2 = A_1^2 + A_2^2 + A_3^2 - \tfrac{1}{5}A^2.$$

If we make a similar construction by drawing CP′ perpendicular to the form OAB, we have also

$$K^2 = \tfrac{1}{10}(l^2 + ln' + m^2);\quad \text{therefore } mn = ln', \text{ or } m : l = n' : n.$$

10838. (Professor NEUBERG.)—On considère les quadriques qui passent par quatre points donnés et dans lesquelles trois diamètres conjugués sont parallèles à des droites données. Trouver (1) le lieu du centre, et (2) le lieu de l'extrémité de l'un de ces diamètres.

Solution by Professor SCHOUTE.

Prenons pour axes de coordonnées les parallèles aux droites données, menées par un des quatre points donnés, et soient x_i, y_i, z_i $(i = 1, 2, 3)$, les coordonnées des trois autres points. Une quadrique pour laquelle les axes coordonnées représentent les directions de trois diamètres conjugués peut être représentée par l'équation

$$Ax^2 + By^2 + Cz^2 + 2Px + 2Qy + 2Rz + S = 0.$$

Donc, le lieu géométrique du centre s'obtient par l'élimination des rapports des coefficients A, B ... R, entre les équations

$Ax_i^2 + By_i^2 + Cz_i^2 + 2Px_i + 2Qy_i + 2Rz_i = 0 \ (i = 1, 2, 3),$

$Ax + P = 0, \quad By + Q = 0,$

$Cz + R = 0,$ dans la forme :—

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & 2x_1 & 2y_1 & 2z_1 \\ x_2^2 & y_2^2 & z_2^2 & 2x_2 & 2y_2 & 2z_2 \\ x_3^2 & y_3^2 & z_3^2 & 2x_3 & 2y_3 & 2z_3 \\ x & 0 & 0 & 1 & 0 & 0 \\ 0 & y & 0 & 0 & 1 & 0 \\ 0 & 0 & z & 0 & 0 & 1 \end{vmatrix} = 0$$

Donc le lieu du centre est une surface du troisième ordre, qui passe par les droites infiniment éloignées des plans coordonnés.

De la même manière on obtient l'équation du lieu des extrémités du diamètre parallèle à l'axe des x en éliminant les mêmes rapports entre les équations

$Ax^2 + By^2 + Cz^2 + 2Px + 2Qy + 2Rz = 0,$

$Ax_i^2 + By_i^2 + Cz_i^2 + 2Px_i + 2Qy_i + 2Rz_i = 0 \ (i = 1, 2, 3),$

$Bx + Q = 0, \quad Cz + R = 0.$

Ainsi l'on trouve la surface :

$$\begin{vmatrix} x^2 & y^2 & z^2 & 2x & 2y & 2z \\ x_1^2 & y_1^2 & z_1^2 & 2x_1 & 2y_1 & 2z_1 \\ x_2^2 & y_2^2 & z_2^2 & 2x_2 & 2y_2 & 2z_2 \\ x_3^2 & y_3^2 & z_3^2 & 2x_3 & 2y_3 & 2z_3 \\ 0 & y & 0 & 0 & 1 & 0 \\ 0 & 0 & z & 0 & 0 & 1 \end{vmatrix} = 0$$

du quatrième ordre, &c.

10654. (Professor DE LONGCHAMPS.)—Résoudre l'équation

$$\frac{1}{x(x-a)(x-b)} + \frac{1}{a(a-x)(a-b)} + \frac{1}{b(b-x)(b-a)} + \frac{1}{ax^2 - abx - a^3} = 0.$$

Solution by H. J. WOODALL.

By taking the fractions two and two according to the factors of the denominators, the question easily resolves. The roots are $\pm a$, a, b.

10511. (R. TUCKER, M.A.)—O, O_a, O_b, O_c are the in- and ex-centres of the triangle ABC; through B, C, lines are drawn parallel to CO_a, CO; BO_a, BO, meeting AC, AB produced in E_1', E_1; $F_1'F_1$, respectively. Draw AD, AD′, cutting BC in D, D′, so that

$$\sin BAD : \sin CAD = AF_1 : AE_1,$$

and
$$\sin BAD' : \sin CAD' = AF_1' : AE_1';$$

then prove that OD′, O_aD pass through the Lemoine-point. [The same holds, of course, for the four lines similarly obtained for the remaining angles.]

Solution by the PROPOSER.

By construction, $AE_1 = a+b$, $AF_1 = a+c$, $AE_1' = a \sim b$, $AF_1' = a \sim c$. Draw the lines AD, AD′, so that

$$\sin BAD : \sin CAD = AF_1 : AE_1, \quad \sin BAD' : \sin CAD' = AF_1' : AE_1'.$$

The equations to OD′, O_aD are

$$\alpha(b-c)+\beta(c-a)+\gamma(a-b) = 0, \quad \alpha(b-c)-\beta(c+a)+\gamma(a+b) = 0.$$

These evidently intersect in K, the "S" point.

Similarly for the other cases.

10735. (R. CHARTRES, M.A.)—A particle projected from one extremity of the horizontal base of a triangle falls at the other extremity, having passed through the orthocentre and the centre of the in-circle of the triangle: prove that the sides of the triangle are in arithmetical progression.

Solution by the PROPOSER.

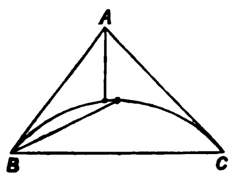
By a well-known problem, if a particle projected from B at $\angle\theta$ with the horizon falls at C after passing through the vertex,

we have $\quad \tan\theta = \tan B + \tan C$;

hence $\quad \tan\frac{1}{2}B + \tan\frac{1}{2}C = \cot C + \cot B$,

or $\quad \sin B + \sin C = 2\sin A$,

or $\quad b + c = 2a$.



10834. (Professor MORLEY, M.A.)—A triangle is circumscribed to a conic so that the normal at any point of contact passes through the opposite vertex. Show that the symmedian point of the triangle is the centre of the conic.

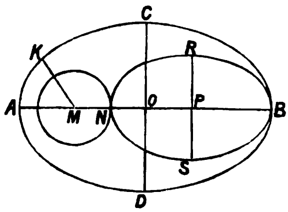
Solution by Professor SCHOUTE; R. TUCKER, M.A.; *and others.*

The three normals intersect in the orthocentre of the circumscribed triangle ABC. The triangle DEF, the vertices of which are the points of contact of the conic with the sides of ABC, is the pedal triangle of ABC. Now the joins of A, B, C with the mid-points of the segments EF, FD, DE pass through the centre of the conic. Moreover, BC and EF are anti-parallel with respect to the angle A; therefore the join of A with the mid-point of EF passes through the symmedian point of ABC, &c.

10836. (Professor MATZ, M.A.)—**From a point taken at random in the left-hand half of the major axis ($=2a$) of an ellipse whose minor axis is unknown, a circle is drawn at random, but so as to lie wholly in the surface of the ellipse. Show that the average area of the ellipse, whose major axis is that portion of the given major axis between its right-hand extremity and the circumference of the circle, is $\dfrac{\pi a^2}{672}\left(\dfrac{2205\pi+2012}{15\pi+17}\right)$.**

Solution by Professor G. B. M. ZERR, M.A.

Let ADBC be the ellipse whose major axis AB $=2$A, and minor axis CD is unknown; M the random point, the centre of the random circle; NSBR the ellipse whose average area is to be determined; BN its major axis, and PR its minor axis.



Let OC $=w$, AM $=x$, MN $=y$, PR $=z$, NP $=\frac{1}{2}(2a-x-y)=v$,

$$\text{MK}=\left\{1-\frac{(a-x)^2}{a^2-w^2}\right\}^{\frac{1}{2}}=y_1,$$

$w^2/a=x_1=$ radius of curvature at A.

The area of NSBR $=\pi vz$; also the limits of z are 0 and v; those of y are 0 and x when x is less than x_1, and 0 and y_1 when x is greater than x_1; those of x are 0 and a, and of w, 0 and a.

Then the required average is

$$\Delta=\frac{\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v \pi vz\,dx\,dy\,dz+\int_{x_1}^a\int_0^{y_1}\int_0^v \pi vz\,dx\,dy\,dz\right]dw}{\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v dx\,dy\,dz+\int_{x_1}^a\int_0^{y_1}\int_0^v dx\,dy\,dz\right]dw}$$

$$=\frac{360}{a^4(15\pi+17)}\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v \pi vz\,dx\,dy\,dz+\int_{x_1}^a\int_0^{y_1}\int_0^v \pi vz\,dx\,dy\,dz\right]dw$$

$=$ the result stated in the Question.

10674. (ELIZABETH BLACKWOOD).—Examine the accuracy of the following construction, given in books on geometrical drawing, for inscribing a regular polygon of n sides in a given circle:—Divide the diameter AB into two parts at C, such that AC : AB $=2:n$. Through P, the vertex of an equilateral triangle on AB, draw a straight line passing through C and meeting the circumference at Q, on the opposite side of the diameter to P. Then AQ is one side of the required polygon.

Solution by H. J. WOODALL.

By ordinary transformations in trigonometry we obtain

$$\sin \text{QOA}=\sqrt{3}\left\{n\sqrt{n^2+16n-32}-(n-4)^2\right\}/\left\{4(n^2-2n+4)\right\}.$$

This is not in general equivalent to $\sin 2\pi/n$ (as can be shown by trial, except when $n=4$), and therefore the construction is not accurate.

10242. (J. C. St. Clair.)—**Given a point P on a circle, and a line *s* which is the Simson-line of any inscribed triangle; prove that (1) the triangle may vary, and find the locus of centroids; and (2) if P vary, *s* remaining fixed, and conversely—determine geometrically the limiting positions of P in the first case, and the limiting envelope of *s* in the second case, so that it may be possible to inscribe an *acute-angled* triangle, having *s* for its Simson-line.**

Solution by the Proposer.

(1) Take any chord PA such that the circle on PA as diameter shall cut the line *s* in two points β, γ. The chords joining A to β, γ are the sides of an inscribed triangle of which *s* is the Simson-line.

Let H be the orthocentre of any such triangle. Then PH is bisected by *s* (Casey's *Sequel to Euclid*, III., 14). Consequently the locus of H is a straight line parallel to *s* at twice its distance from P. And if O, the centre of the circle, be joined to H, the centroid G and also the centre of the nine-points circle lie on OH and divide it in constant ratios. Hence these points lie on lines parallel to *s*.

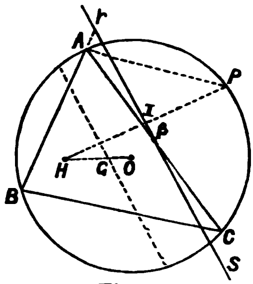


Fig. 1.

(2a) Let *s* be fixed. On the side opposite to O draw a tangent parallel to *s* touching the circle in H (Fig. 2). Draw also a chord RR′ parallel to *s* and equidistant from it. Since the orthocentre of an acute-angled triangle lies within it, the point H is the limiting position for inscribed acute-angled triangles. Hence P must lie on the arc RHR′, since it has been shown that P and the locus of orthocentres are equidistant from *s*.

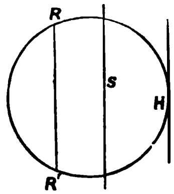


Fig. 2.

(2b) Since, as above, the different points of the circle are the limiting positions of the orthocentres, and the variable line *s* is equidistant from P and the tangent, it is evident that *s* in its limiting position will always touch a circle whose radius is half that of circle O, and which touches the latter at P. If *s* cut this circle, an acute-angled triangle may be inscribed.

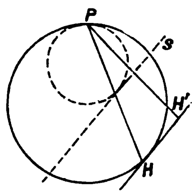
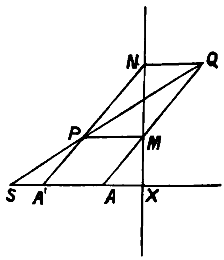


Fig. 3.

6922. (Professor Wolstenholme, M.A.)—Four points S, A′, A, X are taken on a straight line so that SA′ = AX; the point S will be called the

focus, and the straight line through X at right angles to SX the directrix. Then, if from any point P in the plane PM be let fall perpendicular on the directrix, and SP, AM meet in Q, whatever be the locus of P, (1) the locus of Q will be a curve of the same order and class; (2) the tangents at P, Q will always intersect on the directrix; (3) if QN be drawn perpendicular to the directrix, NPA′ is a straight line; (4) if the locus of P be a conic having the given focus and directrix, so also will the locus of Q; (5) if the locus of P be a parabola with S for focus and vertex A′, the locus of Q is a parabola with focus S and vertex A; (6) if the tangents at P, Q include a given angle, the loci of P, Q will be both parabolas with focus S; A′, A will lie upon the tangents at the vertices; and the axes will be equally inclined to SX (or corresponding tangents to two such parabolas).

Solution by the PROPOSER.



Let $SA = a$, $AX = a'$, and let the coordinates of P, Q be (x, y), (X, Y), S being the origin, and SX the axis of x. Then

$$\frac{y}{Y} = \frac{x}{X} = \frac{a'}{X-a} = \frac{x-a'}{a}.$$

Hence (xy), (XY) have a $(1,1)$ correspondence, and, whatever be the locus of P, the locus of Q will be a curve of the same order and class. Also,

$$\frac{dy}{dx} = \frac{dY}{dx}\,\frac{a'}{X-a} - \frac{a'Y\,(dX)/(dx)}{(X-a)^2}$$

$$= \left\{\frac{dY}{dX}\,\frac{a'}{X-a} - \frac{a'Y}{(X-a)^2}\right\} \div \frac{-aa'}{(X-a)^2} = \frac{1}{a}\left\{Y-(X-a)\frac{dY}{dX}\right\}.$$

Again, $$y + (a+a'-x)\frac{dy}{dx} = \frac{aY}{X-a} + \left(a+a'-X\frac{a'X}{X-a}\right)\frac{1}{a}\left\{Y-(X-a)\frac{dY}{dX}\right\}$$

$$= Y + (a+a'-X)\frac{dY}{dX}.$$

Thus the tangents at P, Q to the loci of P, Q intersect upon the straight line XM.

If QN be let fall perpendicular upon XM and NP meet SX in A′,

$$SA' : NQ = SP : PQ = x : X-x,$$

or $$SA' = \frac{x\,(X-a-a')}{X-x} = \frac{a'X}{X-a}\cdot\frac{X-a-a'}{X-(a'X)/(X-a)} = a'.$$

Thus A′ is a fixed point, and P might be constructed from Q in the same way as Q from P.

Again, $$\frac{x^2+y^2}{(a+a'-x)^2} = \frac{X^2+Y^2}{\{(a+a')\,(X-a)/(a')-X\}^2} = \frac{a'^2}{a^2}\,\frac{X^2+Y^2}{(a+a'-X)^2},$$

so that if the locus of P be a conic with focus S, and directrix XMN,

the locus of Q is a conic with the same focus and directrix, but eccentricity altered in the ratio $a : a'$.

If the locus of P be a parabola with focus S and vertex A′,

$$y^2 = 4a'(a'-x), \quad \text{whence} \quad Y^2 = 4(X-a)(X-a-X) = 4a(a-X),$$

or the locus of Q is a parabola with focus S and vertex A.

The tangent of the angle between the tangents at P, Q is

$$\left(\frac{dy}{dx} - \frac{dY}{dX}\right) \div \left(1 + \frac{dy}{dx}\frac{dY}{dX}\right),$$

and, if this $= 1/m$, a constant.

$$\frac{1}{m} = \left[\left\{Y + (a-x)\frac{dY}{dX}\right\} - a\frac{dY}{dX}\right] \Big/ \left[a + \left\{Y + (a-x)\frac{dY}{dX}\right\}\frac{dY}{dX}\right],$$

or

$$Y + (a-X)\frac{dY}{dX} = \left[a\left(1 + m\frac{dY}{dX}\right)\right] \Big/ \left[m - \frac{dY}{dX}\right],$$

which is an equation of Clairaut's form; or the locus of Q is in genera a straight line touching the parabola

$$(Y+mX)^2 = 4(mY-a)(X-a) \quad \text{or} \quad (Y-mX)^2 = 4a(a-X-mY),$$

a parabola whose focus is S and the tangent at whose vertex passes through A. The locus of Q may also be this parabola whose equation is the singular solution.

So the locus of P is in general a tangent to the parabola

$$(y-mx)^2 = 4(x-a')(x-a'-x-my) = 4(a'+my)(a'-x)$$

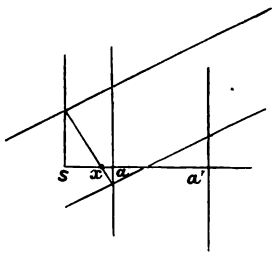
or

$$(y+mx)^2 = 4a'(a'-x+my);$$

whose focus is S and the tangent at the vertex passes through A′. The locus may also be the parabola itself.

If we reciprocate the whole system with respect to S, instead of the construction of the point Q from the point P, we get the following :—

Take $Sxaa'$ in one straight line, so that $Sx = 1/SX$, $Sa = 1/SA$, $Sa' = 1/SA'$ and through S, a, a' draw straight lines at right angles to the former; then, if any straight line p meet the line through S in P, and Px meet the line through a in Q, and we draw through Q a straight line parallel to p, the straight lines p, q are the reciprocals of the points P, Q of the former construction, and since these parallels are at distances from x in the constant ratio $Sx : xa$, the envelopes must always be similar curves; and x being their centre of similarity, their points of contact with the envelopes must lie on a straight line through x. This proves the first two theorems, and the others are also almost immediately obvious in this reciprocal figure. Of course this seems the natural way when once pointed out, but I did not discover it until, after having investigated all the theorems directly as above, I reciprocated the last upon S.



[The last theorem and its reciprocal may be stated as follows :—

(1) Two parabolas have a common focus S, SX is let fall perpendicular

on one of the bisectors of the angles between the directrices, and meets the tangents to the two parabolas parallel to this bisector in A, A′; from a point P on one parabola is let fall PM perpendicular on the bisector, and SP, A′M intersect in Q: then Q will be a point on the second parabola, the tangents at P, Q will intersect on the bisector, and the angle between them will be constant and equal to half the angle between the directrices; also, if QN be let fall perpendicular on the bisector, N, P, A will be collinear.

(2) Two circles, U, V, intersect in a point S, and X is one of their centres of similitude; XS meets the circles again in A, A′, and through S, A, A′ are drawn three straight lines at right angles to XS (s, a, a'); any tangent (p) to U meets s in P, and PX meets a' in Q; the straight line (q) drawn through Q parallel to the tangent to U will be a tangent to V, the points of contact will be collinear with S, the angle between the tangents will be constant and equal to the angle which either tangent at S makes with SX, and if q meet s in P′, XP′, p, and a will (be compunctual) concur in a point.]

9763. (E. W. Rees, B.A.)—If O be the orthocentre, and $2s$ the perimeter of a triangle ABC, and if r_a, r_b, r_c are the radii of the inscribed circles of the triangles OBC, OCA, OAB respectively, prove that

$$(\tan\tfrac{1}{2}B-\tan\tfrac{1}{2}C)/r_a+(\tan\tfrac{1}{2}C-\tan\tfrac{1}{2}A)/r_b+(\tan\tfrac{1}{2}A-\tan\tfrac{1}{2}B)/r_c=0\ \ldots(1),$$

$$2s\,(r_br_c+r_cr_a+r_ar_b)^2=r_ar_br_c\,(s+r_a+r_b+r_c)^2\ \ldots\ldots\ldots\ (2).$$

Solution by the Proposer.

$$r_a=\frac{OB\,.\,OC\sin A}{OB+OC+a}\ \text{ and }\ \frac{OA}{\cos A}=\frac{OB}{\cos B}=\frac{OC}{\cos C}=K=\frac{a}{\sin A},\ \&c.$$

$$=\frac{K\sin A\cos B\cos C}{\cos B+\cos C+\sin A}=\frac{K\sin A\cos B\cos C}{2\sin\frac{1}{2}A\,(\cos\frac{1}{2}B+\sin\frac{1}{2}B)\,(\cos\frac{1}{2}C+\sin\frac{1}{2}C)}$$

$$=\tfrac{1}{2}K\,(\sin A+\sin B+\sin C)\,(1-\tan\tfrac{1}{2}B)\,(1-\tan\tfrac{1}{2}C)$$

$$=\tfrac{1}{2}s\,(1-\tan\tfrac{1}{2}B)\,(1-\tan\tfrac{1}{2}C),\ \text{whence we obtain the equation (1).}$$

Also, $2/s\,\Sigma r_a=3-2\Sigma\tan\frac{1}{2}A+\Sigma\tan\frac{1}{2}B\tan\frac{1}{2}C=2\,(2-\Sigma\tan\frac{1}{2}A)$,

$$4/s^2\,\Sigma r_br_c=(1-\tan\tfrac{1}{2}A)\,(1-\tan\tfrac{1}{2}B)\,(1-\tan\tfrac{1}{2}C)\,\Sigma\,(1-\tan\tfrac{1}{2}A)$$

$$=(8/s^3\,r_ar_br_c)^{\frac{1}{2}}\,(3-\Sigma\tan\tfrac{1}{2}A)=(8/s^3\,r_ar_br_c)^{\frac{1}{2}}\,(s+\Sigma r_a/s),$$

whence we obtain the equation (2).

9472. (W. J. Greenstreet, M.A.)—If the sides of an equilateral triangle, of area Δ, be bent on a sphere of radius r (large compared with sides of the triangle); prove that the area of the spherical triangle is, approximately, $\Delta+\Delta^2/(2\sqrt{3}\,.\,r^2)$.

Solution by W. J. Greenstreet, M.A.; Prof. Aiyar; *and others.*

$\tan^2 \frac{1}{4}E = \tan \frac{1}{2}s \tan \frac{1}{2}(s-a)$; $\frac{1}{4}s = 3a/4r$ where a is side of Δ; therefore, putting $2x = 3a$, $s = x/r$; also

$$\tan \tfrac{1}{2}s = \tfrac{1}{2}s\left(1 - \frac{s^2}{24} + \dots\right) \Big/ \left(1 - \frac{s^2}{8} + \dots\right) = \frac{x}{2r}\left(1 - \frac{x^2}{24r^2}\right) \Big/ \left(1 - \frac{x}{8r^2}\right)$$

$$= \frac{x}{2r}\left(1 - \frac{x^2}{24r^2}\right)\left(1 + \frac{x}{8r^2} + \dots\right) = \frac{x}{2r}\left(1 + \frac{x^2}{12r^2}\right).$$

$$\therefore \quad \tan^2 \tfrac{1}{4}E = \frac{x}{2r} \cdot \left(\frac{x-a}{2r}\right)^3 \left(1 + \frac{x^2}{12r^2}\right)\left(1 + \frac{(x-a)^2}{12r^2}\right)^3;$$

$$\therefore \quad E = \frac{\Delta}{r^2}\left(1 + \frac{3a^2}{24r^2}\right) \quad \text{and} \quad \Delta = \tfrac{1}{4}a^2 \sin 60^\circ, \quad \text{or} \quad a^2 = 4\Delta/\sqrt{3};$$

$$\therefore \quad E = \frac{\Delta}{r^2}\left(1 + \frac{\Delta}{2(3r^2)^{\frac{1}{2}}}\right);$$

whence the stated result follows.

10826. (J. MacNeill.)—In a certain State the tax per £1 on a person's income varies as the square root of the number of pounds, and when the income is £100 the rate per £1 is $6d$. Find the largest net income possible.

Solution by J. C. Horobin, M.A.; D. Biddle; *and others.*

The rate for an income of £1 is $\frac{6}{10}d.$, or £$\frac{1}{400}$. Let I be the gross income; then the tax is $I\sqrt{I} \div 400$, and the net income $N = I\left(1 - \frac{\sqrt{I}}{400}\right)$.
$\frac{dN}{dI} = 1 - \frac{3\sqrt{I}}{800} = 0$, whence $I = \left(\frac{800}{3}\right)^2$. Therefore

$$N = \tfrac{1}{3}I = £23{,}703{\cdot}\dot{7}0\dot{3} = £23{,}703.\ 14s.\ 1d.$$

10855. (M. Molony.)—A man has a cow which at the end of three years produces a female calf, and then brings forth a cow-calf every year afterwards. Each calf, in its turn, brings forth a cow-calf at the end of three years, and one every year subsequently. Show that the owner's stock at the end of twenty years will be 1278.

Solution by Professor Schoute.

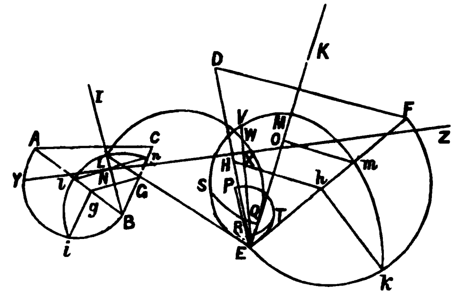
The owner's stock at the end of n years is the $n+1^{\text{th}}$ term of the recurrent series 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28 ...,
the successive terms of which are found by the relation $u_n = u_{n+1} + u_{n-3}$, *i.e.*, the coefficient of x^n in the development of $(1+x-x^3)^{-1}$. Thus the number in question is 1278.

[The limit of the ratio $u_{x+1} : u_x$ for $x = \infty$ is the real root 1·4655... of the equation $x^3 - x^2 - 1 = 0$.]

10715. (J. J. BARNIVILLE.)—**Draw a common bisector to two triangles.**

Solution by D. BIDDLE.

Apparently, every attempt to define the position of the required line algebraically results in an unwieldy biquadratic equation. Probably,



therefore, the question cannot be solved by pure geometry. Nevertheless, it affords an excellent instance of solution by practical methods, which, though founded on pure geometry, are frequently carried out by the aid of postulates not to be found in Euclid.

The line will cut two sides of each triangle. Let B, E be the angles at which such sides severally meet. Bisect them by BI, EK, respectively. From G, H, mid-points of BC, DE, draw Gg, Hh perpendicular to BI, EK. On AB, EF describe semicircles, and draw gi, hk perpendicular to AB, EF. From B, E as centres describe the arcs ilL, kmM. Then Bl, Em are sides of isosceles triangles $= \frac{1}{2}$ABC, $\frac{1}{2}$DEF, respectively; and BN, EO represent their height. Draw EP parallel and equal to BL, and on EM, EP describe semicircles. From M (with radius = EO), and from P (with radius = BN) describe the arcs QS, RT. Join EL, and on it describe a semicircle. Then draw EV so that VW = the portion intercepted between the arcs QS, RT. Make EX = the portion of the line between V and the arc QS, and through X draw YZ at right angles to EV. Then YZ is the line required.

By construction, $Bl^2 = AB \cdot BG$, and $Em^2 = EF \cdot EH$; and these represent the products of the portions of the sides between B, E, respectively, and the required line, YZ, the areas of the triangular segments being $\frac{1}{2}AB \cdot BG \cdot \sin B$ and $\frac{1}{2}EF \cdot EH \cdot \sin E$. Let VEK $= \theta$, whilst DEK $= \alpha$. Then it is easy to see that $EX = (\cos^2\alpha - \sin^2\theta)^{\frac{1}{2}} EM$. For $EO = \cos\alpha \cdot EM$ = radius of arc QS, and $MV = \sin\theta \cdot EM$. Therefore, for any angle θ, the distance from E of a bisector of the triangle DEF, passing through the sides DE, EF, is represented by the intercept between the semicircle MSE and the arc QS. Similarly, the intercept between the semicircle PTE and the arc RT represents the distance from B of a bisector (parallel or identical with the former) of the triangle ABC.

Now, it is evident that VX = that portion of the line EV which lies

between E and the arc QS; and, in like manner, the distance of L from YZ = the portion of EV between E and the arc RT, = WX, since WL is parallel to YZ, both being at right angles to EV. Consequently, by analysis, as by construction, VW = that portion of EV which lies between the arcs QS, RT.

The triangles ABC, DEF, in the case here given, lie on the same side of the line joining the angles at which the intersected sides meet, and VW = the difference of the arc-distances. When they lie on opposite sides of such joining line, VW = the sum of the arc-distances.

10895. (Editor.)—Three conics have a given common directrix, and through the common points of each pair a circle is drawn; prove that the three circles so drawn will be coaxial, and their two common points will be images of each other with respect to the circle which passes through the three foci corresponding to the given directrix.

Solution by Professors Schoute, Wolstenholme, *and others.*

Let r_1, r_2, r_3 be the distances of any point from the three foci corresponding to the given directrix, and let e_1, e_2, e_3 be the three eccentricities. The common points of the conics (2) and (3) will lie on the circle $r_2/e_2 = \pm r_3/e_3$, and the three such circles will have the two common points given by the equations $r_1^2/e_1^2 = r_2^2/e_2^2 = r_3^2/e_3^2$, and have the common radical axis whose equation is

$$e_1^2(r_2^2-r_3^2)+e_2^2(r_3^2-r_1^2)+e_3^2(r_1^2-r_2^2) = 0.$$

And since this equation is satisfied by $r_1^2 = r_2^2 = r_3^2$, the radical axis passes through the circumcentre. Moreover, the circle $r_2/e_2 = \pm r_3/e_3$ has the ends of its diameter along the line x divided harmonically by the circumcircle; hence it cuts the circumcircle orthogonally, and any diameter of the circumcircle will meet it in two points which are mutually images with respect to the circumcircle. Thus the two points common to the three circles, having been proved to lie on a diameter of the circumcircle, must be images with respect to the circumcircle.

[The Question is another form of stating a particular case of the theorem, that the director circles of all conics inscribed in the same quadrilateral have two common points which are images with respect to the circle circumscribing the triangle formed by the diagonals.]

10442. (Professor Wolstenholme, Sc.D.)—The four polar equations

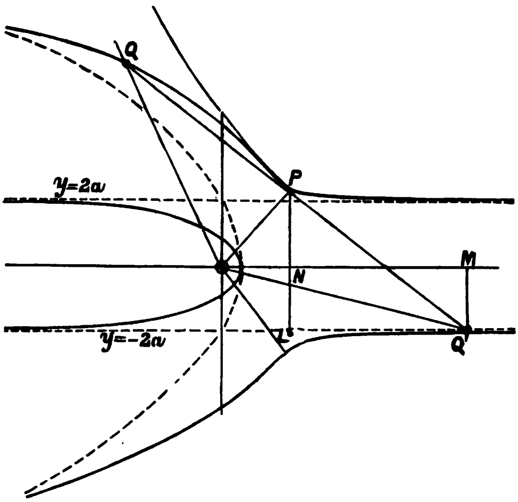
$$r = a/\cos\tfrac{1}{2}\theta\,(\cos\tfrac{1}{2}\theta \pm 1), \quad r = -a/\sin\tfrac{1}{2}\theta\,(\sin\tfrac{1}{2}\theta \pm 1)$$

all represent the same quartic whose equation in rectangular coordinates is

$$y^4+8ay^2(x-3a)+16a^3(a-2x) = 0.$$

Any chord through the origin is divided harmonically by this curve, and the mid-points of the joins of each pair of conjugate points of the harmonic range lie on the parabola whose focus is the origin and latus rectum $4a$. The point P $(2a, 2\sqrt{3}a)$ lies on this curve, and if the tangent at P meet the curve again in Q, Q′, P will bisect QQ′, and, O being the origin, OQ = OQ′ = $12a$; the normal at P passes through the origin, and the radius of curvature at P is $8a$.

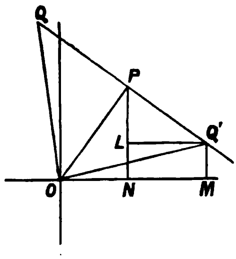
Solution by H. J. WOODALL, Prof. CHAKRIVARTI, *and others.*



The coordinates of any point P may be written either (r, θ), $(-r, \theta+\pi)$, $(r, \theta+2\pi)$, or $(-r, \theta+3\pi)$, and, since we have

$$\cos\tfrac{1}{2}\theta = \sin\tfrac{1}{2}(\theta+\pi) = -\cos\tfrac{1}{2}(\theta+2\pi)$$
$$= -\sin\tfrac{1}{2}(\theta+3\pi),$$

it is quite evident that the four equations refer to the same curve.



To obtain the Cartesian equation, we have

$$a = r\cos^2\tfrac{1}{2}\theta + r\cos\tfrac{1}{2}\theta;$$

and substituting, as usual, for r and θ, we get

$$y^4 + 8ay^2(x-3a) + 16a^3(a-2x) = 0.$$

For one value of θ there will be four values of r: let these be $= \mathrm{OP}_1, \mathrm{OP}_2, \mathrm{OP}_3, \mathrm{OP}_4$. Then, for a H.R., we must have

$$\left(\frac{1}{\mathrm{OP}_1}-\frac{1}{\mathrm{OP}_2}\right)\left(\frac{1}{\mathrm{OP}_3}-\frac{1}{\mathrm{OP}_4}\right) = -\left(\frac{1}{\mathrm{OP}_1}-\frac{1}{\mathrm{OP}_4}\right)\left(\frac{1}{\mathrm{OP}_3}-\frac{1}{\mathrm{OP}_2}\right),$$

which we shall easily find to be true by substituting by means of the four given values of r *in order*, taking OP_1, OP_3 to be the values of r given by $r = a/\cos\frac{1}{2}\theta\,(\cos\frac{1}{2}\theta \pm 1)$. Mid-point of P_1P_3 is

$$r = \tfrac{1}{2}a\left\{1/(\cos^2\tfrac{1}{2}\theta + \cos\tfrac{1}{2}\theta) + 1/(\cos^2\tfrac{1}{2}\theta - \cos\tfrac{1}{2}\theta)\right\},$$

which reduces to a parabola $r = -2a/1-\cos\theta$; $(2a,\ 2\sqrt{3}\,.\,a)$ is on the curve. Tangent at (x, y) to the curve is

$$X\,(8ay^2 - 32a^3) + Y\left\{4y^3 + 16ay\,(x-3a)\right\} = x\,(8ay^2 - 32a^3) + y\left\{4y^3 + 16ay\,(x-3a)\right\};$$

tangent at $(2a,\ 2\sqrt{3}\,.\,a)$ is $X + \sqrt{3}\,.\,Y = 8a$; normal is $X\sqrt{3} = Y$, which passes through $(0, 0)$. Tangents meets curve again at $x = a\,(2 \mp 4\sqrt{6})$, $y = a\,(2\sqrt{3} \pm 4\sqrt{2})$, therefore tangent is bisected at the point of contact;

$$PL = PN - Q'M = 4\sqrt{2}\,.\,a, \quad OP = 4a, \quad LQ' = 4\sqrt{6}\,.\,a,$$

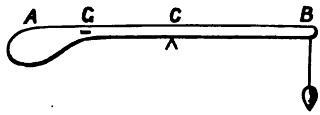
$$Q'P = (PL^2 + LQ'^2) = a\,(32+96)^{\frac{1}{2}} = a\,(128)^{\frac{1}{2}},$$

$$OQ' = a\,(16+128)^{\frac{1}{2}} = 12a = OQ.$$

By substitution in the differential calculus formula for the radius of curvature, it is easy to find $\rho = 8a$.

10891. (Professor ORCHARD, M.A., B.Sc.)—If, in the Danish steelyard, a, b be the distances of the fulcrum from the end at which weights of 10 and 20 lbs., respectively, are suspended, find the distance when 100 lbs. weight is suspended.

Solution by G. E. CRAWFORD, B.A.; J. C. ST. CLAIR; *and others.*



Take moments about B. Thus, if W be the weight of steelyard, and d the distance from B of its c.g. G, $(W+10)a = (W+20)b = (W+100)x$ (where x is the distance we require), because each of these quantities $= Wd$. Hence we have

$$W = \frac{20b - 10a}{a-b}; \quad \therefore \quad x = a\,\frac{W+10}{W+100} = a\,\frac{10b}{90a - 80b} = \frac{ab}{9a-8b}.$$

4337. (By Sir R. S. BALL, F.R.S.).—If in any binary quantic $a_0, a_1, \ldots a_n\,(x, 1)^n \equiv F(x)$, x be changed into

$$\lambda + nF(\lambda)/\left\{nx'F(\lambda) - F'(\lambda)\right\},$$

and the result be cleared of fractions by multiplying it by

$$\left\{nx'F(\lambda) - F'(\lambda)\right\}^n;$$

prove that the coefficient of every power of x' in the expression thus obtained is a covariant of $F(\lambda)$.

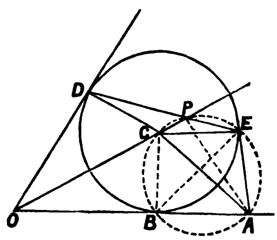
Solution by Professor SEBASTIAN SIRCOM.

Take $a_0 = 1$, and let $F(x) = (x-a_1)(x-a_2)\dots(x-a_n)$; so for $F(\lambda)$.

Making the required substitution and clearing fractions, the first factor becomes $$(\lambda-a_1)\left\{nx'F\lambda-F'(\lambda)+n(\lambda-a_2)\dots(\lambda-a_n)\right\},$$
or $$(\lambda-a_1)\left\{nx'F(\lambda)-(a_2-a_1)(\lambda-a_3)\dots\right.$$
$$\left.\dots(\lambda-a_n)-(a_3-a_1)(\lambda-a_2)(\lambda-a_4)\dots(\lambda-a_n)+\dots\right\}.$$
Multiplying all these together, we have $F(\lambda)$ multiplied by a function of x', each coefficient in which is a function of the differences of the roots and of their differences from λ, each root occurring the same number of times; each coefficient is, therefore, a covariant of $F(\lambda)$.

10894. (Prof. BOURRIENNE.)—Soient XOY un angle fixe et A un point fixe pris sur OX. On trace un cercle quelconque C tangent à OX et en un point D à OY; puis de A on mène à ce cercle une seconde tangente qui le touche en E. Démontrer que la droite DE passe par un point fixe.

Solution by J. C. ST. CLAIR; Prof. SCHOUTE; *and others.*



From A draw AP perpendicular to OC. Then, since the points B, P, E lie on a circle whose diameter is AC, we have

$\angle$ BEP = BAP = BCO = BED.

Hence DE passes through the fixed point P.

10408. (Professor SYLVESTER.)—Solve the Algebraical Conundrum to find a generating function in t, u, such that the coefficient of $t^m \cdot u^n$ when $m+n$ is odd shall be zero, and when $m+n$ is even shall be the half of the greater of the two numbers $m+1$, $n+1$.

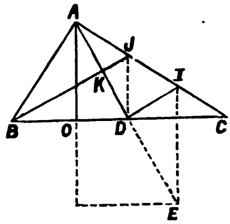
Note by the PROPOSER.

The solution of this conundrum is wanted to complete a theory of mine relating to Hilbert's proof of Gordan's theorem applied to a pair of binary quantics; and it is called a conundrum because its solution seems to depend on intuition, and cannot, without much difficulty, be made to depend on a general method.

The General Function required in the Question is
$$\frac{tu}{(1-t^2)(1-tu)(1-u^2)}.$$

10881. (Professor de LONGCHAMPS.)—Soit yOx un angle droit; sur Oy, on donne un point fixe A, par lequel on mène une transversale mobile rencontrant Ox en C; la bisectrice de OAC coupe Ox en D. Démontrer que (1) la perpendiculaire élevée en D à AD rencontre AC en un point I, dont le lieu géométrique est une parabole, de foyer A; (2) la perpendiculaire menée, par A, à la transversale AC, coupe OX en B, la bisectrice de l'angle ABC rencontre AC en J, le lieu de J est aussi une parabole de foyer A; (3) les droites AD et BJ se coupent en un certain point K; le lieu de E est une droite.

Solution by J. C. ST. CLAIR, Professor SCHOUTE, *and others*.



Produce AD to E, making DE = AD. Then, the triangles ADI, EDI being equal, AI = IE and IE is parallel to AO. Hence, since the locus of E is a straight line parallel to OC, the locus of I is a parabola, focus A.

(2) In the triangle ABD, the angle

$$BAD = C + DAO = C + DAC = BDA.$$

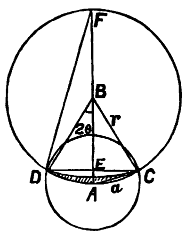
Hence BA = BD and ΔBAJ = BDJ. Therefore AJ = JD, and JDB is a right angle. The locus of J is therefore a parabola, focus A and directrix OC.

(3) Since BJ bisects AD in K, the locus of K is a straight line parallel to OC.

[From the data, we have AK = KD, AJ = JD, AJ = JI; and this proves the three parts of the Question in the order (3), (2), (1).]

10896. (D. BIDDLE.)—Two spheres intersect, and the centre of one lies on the surface of the other. Prove that, when the former sphere is constant, the size of the latter does not affect the area of its surface which is intercepted.

Solution by G. E. CRAWFORD, B.A.; J. C. ST. CLAIR; *and others*.



Let A be the constant sphere, radius a, and let r be the radius of the other sphere. Then portion of area of B intercepted by A

$$= 2\pi r \,.\, AE = \pi AE \,.\, AF$$

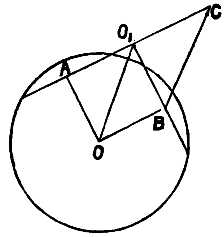
$$= \pi AD^2 = \pi a^2$$

$$= \tfrac{1}{4} \text{ total surface of constant sphere}$$

$$= \text{area of one of its great circles.}$$

10362. (Prof. Coupeau.)—Dans un cercle donné, on considère les cordes des deux arcs interceptés respectivement par les côtés d'un angle droit quelconque et par leur prolongement. Démontrer que la droite joignant le milieu d'une des cordes au symétrique du milieu de l'autre corde par rapport à l'un des côtés de l'angle droit a une longueur constante.

Solution by W. J. Greenstreet; Prof. Chakrivarti; *and others.*

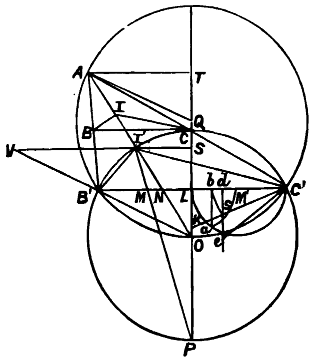


If AO_1B be the right angle, and A, B the middle points of the chords, C the image of A with respect to O_1B, we have AO_1BO and O_1OBC are parallelograms; therefore

$$BC = OO_1 = \text{constant.}$$

8990. (Professor Bordage.)—Construct a triangle, knowing the sum of two sides, the portion of the bisector of the angle formed by the given sides included between the summit and the point of intersection of the bisectors, and the ratio of the same portions of the two other bisectors.

Solution by D. Biddle.



The accompanying diagram combines an analysis of one triangle and the construction of another.

Let I′ be the incentre of AB′C′, and TP the perpendicular bisector of its base. Then Q, on TP, being the circumcentre, AI′ (produced) meets TP in O, at the point of intersection of the circumcircle. Moreover, O is the centre of the circle passing through B′, I′, C′; and if from P, the point of intersection of this circle with TP, PI′ be drawn, it bisects the angle B′I′C′, and

$$B'M : MC' = B'I' : I'C';$$

also $B'N : NC' = AB' : AC'$.

Again, producing OB′, draw I′V parallel to B′C′. Then VI′ = AB′, and VB′ = AI′; also

$$VI' : B'N = AB' : B'N = VO : B'O = VO : I'O,$$

and it follows that $VB' : I'N = AB' : B'N = AI' : I'N$.

Consequently, of the bisector of the angle at the summit, A, the portion

between the summit and the incentre is to that between the incentre and the base, as the sum of the sides to the base.

Now, let B′C′ be the given sum of the sides ($=2l$), C′K = twice the distance from the summit to the incentre ($=2k$), and B′M : MC′ = the given ratio between the lines joining the incentre with the other angles ($=l-m : l+m$, where $m =$ LM). The key to the required construction is the length LP ($=x$). For $\text{OP} = (x^2+l^2)/(2x)$, $\text{OL} = (x^2-l^2)/(2x)$,

Let MPO $=\theta$; then I′OS $=2\theta$,

$$\sin\theta = m/(x^2+m^2)^{\frac{1}{2}}, \quad \cos 2\theta = 1-2\sin^2\theta = (x^2-m^2)/(x^2+m^2).$$

Now $$\text{LS} = \text{OP}\,(1+\cos 2\theta) - x = x\,(l^2-m^2)/(x^2+m^2).$$

But, from the preceding analysis, it is clear that

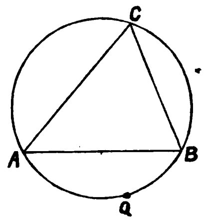
$$\text{NI}' = k = \text{LS}\,.\sec 2\theta = x\,(l^2-m^2)/(x^2-m^2),$$

whence $$x = \left(\frac{l^2-m^2}{2k}\right) \pm \left\{\left(\frac{l^2-m^2}{2k}\right)^2 + m^2\right\}^{\frac{1}{2}}.$$

The following construction is therefore valid. On LC′ ($=l$, taken as unity) describe a semicircle, and make $\text{L}a = \text{LM} = m$; draw ab perpendicular to LC′, and on C′K make C′s = C′b. Draw sd perpendicular to LC′, and produce to e, making $de =$ LM. Join C′e, and make LP = C′e + C′d. Find O, the perpendicular from which bisects PB′; and describe the circle PB′C′. Through M draw PI′, to meet this circle in I′, and join B′I′, C′I, and OI′, the last intersecting B′C′ in N. Anywhere on OI′, take AI = ½C′K = k, and from I draw IB, IC parallel to I′B′, I′C′. Finally, make AB = B′N, AC = NC′, and join BC. ABC is the required triangle.

10890. (Professor PURSER.)—If the tangents t_1, t_2, t_3, drawn to a circle S from the vertices of a triangle, are such that the sum of two of the rectangles at_1, bt_2, ct_3 is equal to the third; prove that S touches the circumcircle of the triangle, without assuming that S and the circumcircle have a real limiting point.

Solution by the PROPOSER.



Let $at_1+bt_2 = ct_3$. Divide the arc AB (that arc which does not contain C) in Q, so that

the chord AQ : chord BQ $= t_1 : t_2$.

Then, by PTOLEMY's Theorem,

$$a\,.\,\text{AQ} + b\,.\,\text{BQ} = c\,.\,\text{CQ};$$

$$\therefore\ \frac{t_1}{\text{AQ}} = \frac{t_2}{\text{BQ}} = \frac{at_1+bt_2}{a\,.\,\text{AQ}+b\,.\,\text{BQ}} = \frac{c\,.\,t_3}{c\,.\,\text{CQ}} = \frac{t_3}{\text{CQ}};$$

∴ A, B, C lie on a circle coaxal with S and the point-circle Q.

Therefore Q is a limiting point of S and the circumcircle, and, since Q lies on the latter, these circles touch.

[If we take the triangle of the enunciation as that of reference, and let $px+qy+rz=0$ be the radical axis of the circle S, and the circumcircle $a^2yz+b^2zx+c^2xy=0$, we shall have $p/t_1^2=q/t_2^2=r/t_3^2$. Now S will touch the circumcircle if the radical axis touch it, the condition for which is

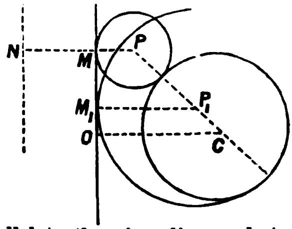
$$a^4p^2+b^4q^2+c^4r^2 = 2b^2c^2qr+2c^2a^2rp+2a^2b^2pq,$$

or $\quad (at_1+bt_2+ct_3)(-at_1+bt_2+ct_3)(at_1-bt_2+ct_3)(at_1+bt_2-ct_3)=0$;

i.e., if two of the three at_1, bt_2, ct_3 be together equal to the third.]

10910. (Rev. T. Roach, M.A.)—Find the locus of the centre of a circle which touches a circle and a straight line.

Solution by J. C. St. Clair; B. W. Mainprise; *and others.*



Let C be the centre of the fixed circle (radius $= r$), and P that of the variable one. Draw PM perpendicular to the given line. Then, evidently,

$$\text{CP} = \text{PM} \pm r,$$

according as the contact of the circles is external or internal.

(1) If the line does not cut the circle C, the locus of P will be two parabolas, similarly placed, having C for their focus, and for directrices two lines parallel to the given line, and at a distance r on each side of it.

(2) If the line touch C in M_1, the centres of internal contact lie on the line M_1C.

(3) If the line cut the circle C, the locus consists of two parabolas oppositely placed; the locus for external contact on one side of the line, and of internal contact on the other side, being the same curve.

10478. (Professor Catalan.)—Si l'on a $f+g+h=1$, l'égalité

$$abc\left\{\frac{f}{a^2}+\frac{g}{b^2}+\frac{h}{c^2}-\left(\frac{f}{a}+\frac{g}{b}+\frac{h}{c}\right)^2\Sigma\frac{a(b-c)^2}{f}\right.$$
$$=\left\{(af+bg+ch)\left(\frac{f}{a}+\frac{g}{b}+\frac{h}{c}\right)-1\right\}\Sigma\frac{a^2(b-c)^2}{f}$$

est une identité.

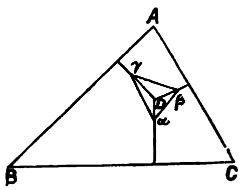
Solution by H. J. Woodall.

Making each side homogeneous by means of the relation $f+g+h=1$, collecting the terms into groups according to f, g, h, the given equation easily reduces to $\;fgh\left[\Sigma\left\{a^2(b-c)^2/f\right\}\right]\times\left[\Sigma\left\{a(b-c)^2/f\right\}\right]/abc.$

[The solution of problems like the above is greatly assisted by the symmetry (which already exists as proposed) and homogeneity (which exists with respect to a, b, c, and can be obtained with respect to f, g, h by means of the given relation). These two conditions being given, the only thing which remains is to collect into groups of sums (Σ) or products (Π)].

10765. (Professor NEUBERG.)—A, B, C, D étant quatre points d'un même plan, si quatre forces parallèles appliquées en ces points se font équilibre, l'équilibre a encore lieu après qu'on a transporté chaque force au centre du cercle circonscrit au triangle qui a pour sommets les points d'application des trois autres forces.

Solution by Professors RAMASWAMI AIYAR, SARKAR, *and others.*



Suppose A′, B′, C′, D′ are the centres; and let $\alpha\beta\gamma$ be the polar of the triangle ABC with respect to a circle whose centre is D and radius κ.

Now, in the quadrilaterals $\alpha\beta\gamma$D, A′B C′D′, *all* the corresponding sides are parallel; for example, $\alpha\beta$ and A′B′ are parallel, both of them being perpendicular to CD, the former because it is the polar of C, and the latter because A′ and B′ are both on the perpendicular bisector of CD. Hence $\alpha\beta\gamma$D and A′B′C′D′ are similar. ...... (1).

Again, if x, y, z be the perpendiculars from D on the sides of the ΔABC,

$$\frac{\Delta \mathrm{DBC}}{\Delta \mathrm{D}\beta\gamma} = \frac{a \,.\, x}{(\kappa^2/y)\,(\kappa^2/z)\sin \mathrm{A}} = \frac{2\mathrm{R}xyz}{\kappa^4}; \;\; \therefore \; \frac{\Delta \mathrm{DBC}}{\Delta \mathrm{D}\beta\gamma} = \frac{\Delta \mathrm{DCA}}{\Delta \mathrm{D}\gamma\alpha} = \frac{\Delta \mathrm{DAB}}{\Delta \mathrm{D}\alpha\beta} \;...(2).$$

But, by statics, the numerators are proportional to the forces acting at A, B, C; these may, therefore, be transferred to α, β, γ without affecting the equilibrium. But A′B′C′D′ is similar to $\alpha\beta\gamma$D. Hence the theorem.

10721 & 10753. (Professor LAMPE, LL.D.)—Solve the equations

$$2^{\frac{1}{2}(n-1)} \cos x \cos 2x \cos 3x \ldots \cos \tfrac{1}{2}(n-1)\,x = 1,$$

$$2^{\frac{1}{2}(n-1)} \sin x \sin 2x \sin 3x \ldots \sin \tfrac{1}{2}(n-1)\,x = n^{\frac{1}{2}}.$$

Solution by Professors LAMPE, BUSSELL, ZERR, *and others.*

These Questions, and Question 9187 (solved on p. 124 of Vol. XLVIII.), are special cases of the formulæ

$$2^{\frac{1}{2}(n-1)} \sin \pi/n \,.\, \sin 2\pi/n \,.\, \sin 3\pi/n \ldots \sin \tfrac{1}{2}(n-1)\,\pi/n = n^{\frac{1}{2}},$$

$$2^{\frac{1}{2}(n-1)} \cos \pi/n \,.\, \cos 2\pi/n \,.\, \cos 3\pi/n \ldots \cos \tfrac{1}{2}(n-1)\,\pi/n = 1$$

(n being odd), given by Euler in a more general form (*Introductio in Analysin*, Chap. XIV., §§ 240–245) as ensuing from the expressions for $\cos nz$ and $\sin nz$.

Inversely, we may ask for all angles satisfying the equations in the above Questions 10721 and 10753.

Putting $n = 9$, we have

$$16 \sin x \,.\, \sin 2x \,.\, \sin 3x \,.\, \sin 4x = 3,$$
$$16 \cos x \,.\, \cos 2x \,.\, \cos 3x \,.\, \cos 4x = 1 \quad \text{..............} \quad (1, 2).$$

In order to solve (1), writing y for $\sin x$, and transforming (1) into powers of $\sin x$, we get

$$1024y^{10} - 2304y^8 + 1664y^6 - 384y^4 + 3 = 0,$$

whereas $\quad \sin 9x = y\,(256y^8 - 576y^6 + 432y^4 - 120y^2 + 9) \quad \text{........} \quad (3, 4).$

The polynomials (3), (4) have the common factor $64y^6 - 96y^4 + 36y^2 - 3$, vanishing for $x = \pi/9,\ 2\pi/9,\ 4\pi/9$. Separating it from (3), we get the other factor $16y^4 - 12y^2 - 1 = 0$. This equation furnishes

$$y = \sin x = \pm \tfrac{1}{4}\left\{6 + (52)^{\frac{1}{2}}\right\}^{\frac{1}{2}},$$

or $x = 65°\ 19'\ 23''$, &c. Applying the same process to (2), we obtain $\cos x = \pm \cos 36°$, $\pm \cos 72°$ (besides 20°, 40°, 80°, &c.).

The equation $64 \sin^2 x \,.\, \sin^2 2x \,.\, \sin^2 3x = 7$ possesses all multiples of $n(\pi/7)$ (except $n = 0$, mod. 7) as roots, and besides $x = \pm 72° 13' 43''$, &c.

The equation $32 \cos x \,.\, \cos 2x \,.\, \cos 3x \,.\, \cos 4x \,.\, \cos 5x = 1$ has all odd multiples of $\pi/11$ as roots (except those of π). Moreover, it is satisfied by $\pm x = 49° 53' 8''$, $69° 28' 45''$, $123° 33' 4''$, $137° 24' 7''$, &c.

[Professor BUSSELL remarks that the equation resolves itself into

$$\sin 6x \sin 8x \sin 10x \ldots \sin (n-1)\,x$$
$$= \sin x \sin 3x \sin 5x \ldots \sin (\text{prime multiple of})\ x,$$

which can be obtained by multiplying both sides by $\sin x \sin 3x \sin 5x$, &c. in (2). This would simplify the solution for a small number, as $n = 9$ giving $\sin 9x \sin 5x = 0$—*i.e.*, $x = 20°, 40°, 80°$, &c., or $= 36°, 72°$, &c.—but for higher numbers would require little further trigonometrical manipulation.]

10690. (Professor LAUVERNAY.)—L'équation

$$3x^2\,(a + b + c) + 4x\,(ab + bc + ca) + 4abc = 0$$

a ses racines réelles, quels que soient a, b, c. Montrer qu'elles sont rationnelles quand on suppose (1) $b = c$, (2) $2bc = a\,(b + c)$.

Solution by J. D. H. DICKSON ; H. J. WOODALL ; *and others.*

The first part is evident, as the expression under the root is

$$\tfrac{1}{2}\left\{(a-b)^2c^2 + (b-c)^2a^2 + (c-a)^2b^2\right\};$$

whence obviously, if $b = c$, the roots are rational. But this is not the case if $2bc = a\,(b + c)$, for the irrational part contains $\sqrt{3}$, unless there is a further relation between a, b, c.

10206. (W. J. Greenstreet, M.A.)—**Triangles of maximum area are inscribed in an ellipse; find (1) the locus of their orthocentres; (2) the locus of their circumcentres; (3) the envelope of the polar of the centre of the ellipse with respect to the circumcircles; also (4) the same in the case of triangles of minimum area circumscribed to an ellipse.**

Solution by W. J. Greenstreet; Professor Mukhopadhyay; *and others*

If α, β, γ be eccentric angles of vertices of in-triangle, we have for ortho- and circum-centres,

$$x_1 = \frac{a^2+b^2}{2a}\Sigma\cos\alpha - \frac{c^2}{2a}\cos(\alpha+\beta+\gamma), \quad x_2 = \frac{c^2}{4a}\left[\Sigma\cos\alpha + \cos(\alpha+\beta+\gamma)\right],$$

$$y_1 = \frac{a^2+b^2}{2b}\Sigma\sin\alpha - \frac{-c^2}{2b}\sin(\alpha+\beta+\gamma), \quad y_2 = \frac{-c^2}{4b}\left[\Sigma\sin\alpha - \sin(\alpha+\beta+\gamma)\right];$$

for area of $\alpha\beta\gamma$, $\frac{1}{2}(ab)\Sigma\sin(\beta-\gamma)$; hence for maximum ($\alpha$ being constant),

$$\beta = \alpha + \tfrac{2}{3}\pi, \quad \gamma = \alpha - \tfrac{2}{3}\pi.$$

Now, the side $\beta\gamma$ in the maximum triangle will be parallel to the tangent at α, and will cut the diameter through α in $(-\frac{1}{2}a\cos\alpha,\ -\frac{1}{2}b\sin\alpha)$; the centroid of the triangle is the centre of the ellipse, the area is $\frac{3}{4}ab\sqrt{3}$, the ortho- and circum-centres become

$$\left(-\frac{c^2}{2a}\cos 3\alpha,\quad -\frac{c^2}{2b}\sin 3\alpha\right), \quad \left(\frac{c^2}{4a}\cos 3\alpha,\quad \frac{c^2}{4b}\sin 3\alpha\right);$$

$\therefore$ loci required are $a^2x^2+b^2y^2 = \frac{1}{4}c^2$, $a^2x^2+b^2y^2 = \frac{1}{16}c^2$. Hence the envelope of the polar of the ellipse with respect to the circumcircles will be found to be $x^2/a^2+y^2/b^2 = 4(a^2+b^2)/c^2$.

Again, if the points of contact of the circum-triangle be α, β, γ, the vertices are $a\dfrac{\cos\frac{1}{2}\overparen{\beta+\gamma}}{\cos\frac{1}{2}\overparen{\beta-\gamma}}$, $b\dfrac{\sin\frac{1}{2}\overparen{\beta+\gamma}}{\cos\frac{1}{2}\overparen{\beta-\gamma}}$, &c. $\therefore$ for orthocentre

$$4ax_1\cos\tfrac{1}{2}\overparen{\beta-\gamma}\cos\tfrac{1}{2}\overparen{\gamma-\alpha}\cos\tfrac{1}{2}\overparen{\alpha-\beta} = 2(a^2+b^2)\cos\alpha\cos\beta\cos\gamma + a^2\left[\Sigma\cos\alpha - \cos(\alpha+\beta+\gamma)\right],$$

$$4by_1\cos\tfrac{1}{2}\overparen{\beta-\gamma}\cos\tfrac{1}{2}\overparen{\gamma-\alpha}\cos\tfrac{1}{2}\overparen{\alpha-\beta} = 2(a^2+b^2)\sin\alpha\sin\beta\sin\gamma + b^2\left[\Sigma\sin\alpha + \sin(\alpha+\beta+\gamma)\right];$$

for circumcentre and centroid

$$4ax_2\cos\tfrac{1}{2}\overparen{\beta-\gamma}\cos\tfrac{1}{2}\overparen{\gamma-\alpha}\cos\tfrac{1}{2}\overparen{\alpha-\beta} = a^2(\Sigma\cos\alpha + \cos\alpha\cos\beta\cos\gamma) - b^2\cos\alpha\cos\beta\cos\gamma,$$

$$4by_2\cos\tfrac{1}{2}\overparen{\beta-\gamma}\cos\tfrac{1}{2}\overparen{\gamma-\alpha}\cos\tfrac{1}{2}\overparen{\alpha-\beta} = b^2(\Sigma\sin\alpha + \sin\alpha\sin\beta\sin\gamma) - a^2\sin\alpha\sin\beta\sin\gamma;$$

$$12x_3\cos\tfrac{1}{2}\overparen{\beta-\gamma}\cos\tfrac{1}{2}\overparen{\gamma-\alpha}\cos\tfrac{1}{2}\overparen{\alpha-\beta} = a\left[3\Sigma\cos\alpha + \Sigma\cos(\beta+\gamma-\alpha)\right],$$

$$12y_3\cos\tfrac{1}{2}\overparen{\beta-\gamma}\cos\tfrac{1}{2}\overparen{\gamma-\alpha}\cos\tfrac{1}{2}\overparen{\alpha-\beta} = b\left[3\Sigma\sin\alpha + \Sigma\sin(\beta+\gamma-\alpha)\right].$$

The area of the triangle is $-ab\tan\frac{1}{2}\widehat{\beta-\gamma}\tan\frac{1}{2}\widehat{\gamma-\alpha}\tan\frac{1}{2}\widehat{\alpha-\beta}$, a minimum when $\beta = \alpha+\frac{2}{3}\pi$, $\gamma = \alpha-\frac{2}{3}\pi$.

Triangles of minimum area have their sides tangents at the vertices of the triangles of maximum area: their area is $3\sqrt{3}\,ab$; their centroid is the centre of the curve; $\therefore$ the loci of (x_1, y_1), (x_2, y_2) are

$$a^2x^2+b^2y^2 = c^4, \quad a^2x^2+b^2y^2 = \tfrac{1}{4}c^4.$$

The envelope of the polars of the centre of the ellipse with respect to the circumcircles is $\quad x^2/a^2+y^2/b^2 = 16\,(a^2+b^2)^2/c^4$;

and we notice the vertices of the minima triangles lie on the ellipse

$$x^2/a^2+y^2/b^2 = 4.$$

4305. (J. Griffiths, M.A.)—If T denote a variable triangle inscribed in a given triangle ABC so as to be homologous with it, H being the centre of homology, and δ, δ_1, δ_2, δ_3 the centres of the four circles which touch the sides of T; prove that the IV. P circles of the different triangles that can be formed from H, δ, δ_1, δ_2, δ_3 intersect in a point R, and that the locus of R is the IV. P circle of the triangle ABC.

Solution by the Proposer.

The theorem depends upon the following propositions:—

(1) The locus of the centre of an equilateral hyperbola circumscribing a triangle ABC is the IV. P circle of the triangle.

(2) The points A, B, C, H, δ, δ_1, δ_2, δ_3 all lie on the same equilateral hyperbola, whose trilinear equation is

$$(m\cos B-n\cos C)\,l\beta\gamma+(n\cos C-l\cos A)\,m\gamma\alpha+(l\cos A-m\cos B)\,n\alpha\beta = 0,$$

where (l, m, n) denotes the point H, and ABC is the triangle of reference.

10454. (Professor Catalan.)—Si m est impair, transformer

$$\cos^{m-1}x\sin x \text{ en } A_1\sin x+A_3\sin 3x+\ldots+A_m\sin mx.$$

Quelles sont les valeurs des coefficients?

Solution by H. J. Woodall; Professor Zerr; *and others.*

$$2^{m-1}\cos^{m-1}x = 2\cos(m-1)\,x+(m-1)\,2\cos(m-3)\,x + \ldots + \frac{(m-1)!}{[\frac{1}{2}(m-1)]!\,[\frac{1}{2}(m-1)]!},$$

$$2^{m-1}\cos^{m-1}x\,.\sin x = \{\sin mx-\sin(m-2)\,x\}$$
$$+(m-1)\{\sin(m-2)\,x-\sin(m-4)\,x\}+\ldots+\frac{(m-1)!}{[\frac{1}{2}(m-1)]!\,[\frac{1}{2}(m-1)]!};$$

therefore $\cos^{m-1} x \cdot \sin x$

$$= \frac{1}{2^{m-1}} \Big[\sin mx + (m-2)\sin(m-2)x + \frac{(m-1)(m-3)}{1.2}\sin(m-4)x + \dots$$

$$+ \dots + \frac{(m-1)(m-2)\dots(m-\tfrac{1}{2}k+1)(m-k)}{1.2\dots\tfrac{1}{2}k} \times \sin(m-k)x$$

$$+ \dots + \frac{(m-1)!}{[\tfrac{1}{2}(m-1)]!\,\tfrac{1}{2}[(m-1)]!}\Big],$$

$$A_{m-k} = \frac{(m-1)(m-2)\dots(m-\tfrac{1}{2}k+1)(m-k)}{\tfrac{1}{2}k!\times 2^{m-1}}.$$

10603. (Professor MANNHEIM.)—Quelle est, (1) parmi les normales à une ellipse donnée, celle qui est la plus éloignée du centre de cette courbe? (2) Même question pour un ellipsoide.

Solution by Profs. ANDERSON, PROMATHANATHA DATA, *and others.*

1. Let δ be the distance of the normal from the centre of the ellipse. Then, since $\delta^2 = r^2 - p^2$, we must have, for the maximum value of δ, $r\dfrac{dr}{dp} = p$ or $\rho = p$, ρ being the radius of curvature. Hence, if r' be the semi-diameter conjugate to r, $r' = p = (ab)^{\frac{1}{2}}$, and we easily find $\delta = a - b$.

2. Let the equation of the ellipsoid be $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. As before, we must have $r^2 - p^2$ maximum, and therefore $rdr - pdp = 0$, which becomes $\quad x(1 + p^4/a^4)\,dx + y(1 + p^4/b^4)\,dy + z(1 + p^4/c^4)\,dz = 0.$

We have also $\quad xdx/a^2 + ydy/b^2 + zdz/c^2 = 0,$

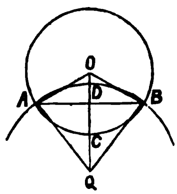
and these are the only equations of condition which dx, dy, dz must satisfy. Hence, λ being an arbitrary multiplier,

$$x(1 + p^4/a^4) + \lambda x/a^2 = 0, \quad y(1 + p^4/b^4) + \lambda y/b^2 = 0, \quad z(1 + p^4/c^4) + \lambda z/c^2 = 0.$$

It is impossible to satisfy these equations by any value of λ, unless either $a = b = c$, or one of the variables x, y, z vanishes. In the latter case, the question reduces to finding the maximum values of δ for the principal elliptic sections. Hence, if $a > b > c$, the maximum value of δ is the greatest of the quantities $a - b$, $b - c$, $a - c$.

10705. (S. TEBAY, B.A.)—Find the radius of a circle which trisects the *area* of a given circle, and cuts the circumference at right angles. [The analagous problem for the trisection of the *circumference* has been solved under Quest. 9447.]

Solution by the PROPOSER.



Let O be the centre of the given circle, Q the centre of the circle required.

Let $OA = a$, $QA = r$, $\angle AOB = 2\theta$. Then

$$r = a \tan\theta,$$

$$\text{area of seg. ACB} = a^2\theta - \tfrac{1}{2}a^2 \sin 2\theta = a^2(\theta - \tfrac{1}{2}\sin 2\theta),$$

$$\text{area of seg. ADB} = r^2(\tfrac{1}{2}\pi - \theta) - \tfrac{1}{2}r^2 \sin 2\theta = a^2 \tan^2\theta\,(\tfrac{1}{2}\pi - \theta - \tfrac{1}{2}\sin 2\theta).$$

Hence $$a^2(\theta - \tfrac{1}{2}\sin 2\theta) + a^2\tan^2\theta\,(\tfrac{1}{2}\pi - \theta - \tfrac{1}{2}\sin 2\theta) = \tfrac{1}{6}\pi a^2,$$

or $$\sin 2\theta + (2\cdot618 - 2\theta)\cos 2\theta = \cdot5236.$$

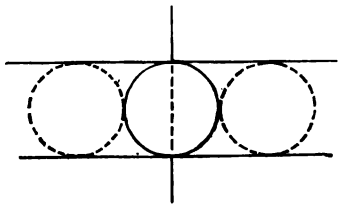
If θ' be the approximate value of θ, and e the error, the correction is

$$d\theta' = -e \,/\, \{2 \sin 2\theta' (2\cdot618 - 2\theta')\}.$$

On trial, it is found that θ lies between 67° and 68°. Taking $\theta = 67°$, we find $e = \cdot0017$, and $d\theta' = \cdot00423$ arc $= 14' + \ldots$; $\therefore\ \theta = 67° 14' + \ldots$. A second approximation makes $\theta = 67° 14' 53''$; $\therefore\ r = 2\cdot3828a$.

10782. (G. F. HOWSE, M.A.)—If three circles be drawn touching each pair at their common point, aud cutting the third at right angles, prove that these will be coaxal.

Solution by the PROPOSER.



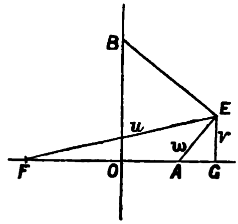
Invent weight to any one of the three points of contact. Two of the circles become parallel lines, the third a circle touching them; the common orthogonal circle the dotted line in the figure, the touching circles the dotted ones in the figure. The rest is obvious, since the dotted line cuts the dotted circles in the same two points.

In exactly the same manner we may show that each of the three orthogonal circles cuts the two circles, touching all three at right angles. Hence they are coaxal.

10893. (Professor WOLSTENHOLME, Sc.D.)—A circle is drawn having double contact with one of a system of confocal ellipses, and touching the minor axis at the centre; prove that (1) the locus of the points of

contact is a lemniscate of Bernoulli, having its foci at the given foci; and (2) the same property is true for confocal hyperbolas, the circles touching the major axis.

Solution by Professors SCHOUTE, BORDAGE, *and others.*



(1) Let F, G be the common foci of the ellipses, and E a point of the first locus. Then the bisector EA of the angle FEG meets FG in a point A for which OA and AE are equal. Now the relations

$$AE^2 = FE \,.\, EG - FA \,.\, AG, \quad OA = AE,$$

give immediately $uw = c^2$, which proves the theorem.

(2) When BE is perpendicular to AE, B will be the centre of a circle touching the hyperbola through E in that point, and the join FG of its foci in O. Therefore the second locus coincides with the first.

[The equation of a circle touching the minor axis, and having double contact with the ellipse $x^2/a^2 + y^2/b^2 = 1$, must be

$$x^2/a^2 + y^2/b^2 - 1 + (1/b^2 - 1/a^2)(x - X)^2 = 0,$$

with the condition $1 = (1/b^2 - 1/a^2)X^2$, or $b^2 = e^2X^2$; so that

$$a^2 - e^2X^2 = a^2 - b^2 = c^2,$$

or, if S, S′ be the given foci, and P the point of contact,

$$SP \,.\, S'P = SO^2.$$

For the hyperbola $x^2/a^2 - y^2/b^2 = 1$, the equation of a circle touching the major axis, and having double contact, will be

$$x^2/a^2 - y^2/b^2 - 1 + (1/a^2 + 1/b^2)(y - Y)^2 = 0,$$

with the condition $1 = Y^2(1/a^2 + 1/b^2)$; so that, if (XY) be a point of contact, $X^2/a^2 = 1 + Y^2/b^2 = 1/e^2 + 1$, and $e^2X^2 - a^2 = a^2e^2$, or $SP \,.\, S'P = SO^2$, as in the former case.]

10618. (H. BROCARD.)—Trouver le lieu du point de contact de deux séries de circonférences, les unes tangentes à deux droites rectangulaires OX, OY, les autres tangentes à OX en un point fixe A.

Solution by Prof. ANDERSON, M.A.; H. J. WOODALL; *and others.*

The differential equations of the system of circles which touch OX at the point A (the distance OA being a), and of the series of circles touching the axes of coordinates, are

$$p\left[(x-a)^2 - y^2\right] - 2y(x-a) = 0, \quad (px - y)^4 - 4x^2y^2(1+p^2)^2 = 0.$$

Eliminating p, we get for the equation of the locus

$$y^4(x^2 + y^2 - a^2)^4 - 4x^2y^2\left[(x-a)^2 + y^2\right]^4 = 0.$$

10914. (Professor SYLVESTER.)—$p_1, p_2, \ldots p_i$ are any i numbers relatively prime to one another, whose product is V

$$\rho_{1,1}, \rho_{1,2}, \ldots \rho_1, \theta_1; \quad \rho_{2,1}, \rho_{2.2} \ldots \rho_2, \theta_2; \quad \ldots \rho_{i,1}, \rho_{i,2}, \ldots \rho_i, \theta_i,$$

are i sets of numbers all less than V. No two ρ's in the same rth set are congruous to each other to the modulus p_r. Out of the natural sequence of numbers, V + 1, V + 2, V + 3, ... jV, all numbers are to be elided which differ by a multiple of p^r from any one of the numbers in the rth set; prove that the number of numbers remaining over after elision is independent of the values of the ρ's in the i sets, and is equal to

$$\left(1-\frac{\theta_1}{p_1}\right)\left(1-\frac{\theta_2}{p_2}\right) \ldots\ldots \left(1-\frac{\theta_i}{p_i}\right)(j-1)\,\mathrm{V}.$$

Solution by H. J. WOODALL.

Consider the effect of $\rho_{r,1}$. Then the numbers removed will be

$$\rho_{r,1}+t_1 p_r, \quad \rho_{r,1}+(t_1+1)p_r, \ldots\ldots \rho_{r,1}+s_1, p_r,$$

and the number of numbers removed is s_1-t_1+1. But, evidently, $(s_1-t_1+1)\,pr = \mathrm{V}(j-1)$; hence the number becomes $\mathrm{V}(j-1)\,p_r$. Taking into account the rth series, this becomes $\mathrm{V}(j-1)\,\dfrac{\theta_r}{p_r}$, and thus the number left is $\mathrm{V}(j-1)\left(1-\dfrac{\theta_r}{p_r}\right)$. This can be continued for the whole i series, whence we have that the number of numbers left is equal to

$$\left(1-\frac{\theta_1}{p_1}\right)\left(1-\frac{\theta_2}{p_2}\right) \ldots\ldots \left(1-\frac{\theta_i}{p_i}\right)(j-1)\,\mathrm{V}.$$

Obviously, any two of these operations may be partly superposable, that is to say, $\rho_{r,k}+t_p p_r$ may equal $\rho_{s,l}+t_q p_s$. Hence we must suppose that "no two ρ's in series are congruous to each other to their respective moduli within the limit V and jV."

10927. (Professor ZERR.)—Suppose the earth an airless homogeneous sphere with an opening from pole to pole. If a marble fall from a distance equal to twice the radius through the centre, find with what velocity it will pass the centre and in what time it will return to the point of starting.

Solution by D. BIDDLE.

Referring to the Solution of 10119 (Vol. LII., p. 37), out of which the present question seems to have arisen, we have R = 20902410 feet, and f (at the earth's surface) = 32·10614 = R/651041. So long as the marble is external to the earth, the attractive force exerted upon it is inversely as the square of its distance from the centre; but when it is internal to the earth, the attractive force exerted upon it, as shown in the solution above

referred to, is directly as its distance from the centre. Taking R = unity, we obtain the following integral:

$$\int_0^v z\,ds = f\int_0^1 \frac{dx}{(2-x)^2} + f\int_1^2 (2-x)\,dx,$$

whence $$v^2 = 2f\left\{(1-\tfrac{1}{2}) + (2-2+\tfrac{1}{2})\right\} = 2f;$$

therefore $$V = (2fR)^{\frac{1}{2}} = 36635{\cdot}9 \text{ feet}.$$

Now, the time occupied in falling through an infinitesimal distance is inversely proportional to the velocity. The velocity of the marble at any part of its fall before reaching the earth is given by

$$v^2 = 2f\int_0^s \frac{dx}{(2-x)^2} = \frac{fs}{2-s}.$$

Consequently, the time occupied in reaching the earth's surface is

$$t = \int_0^1 \left(\frac{2-x}{fx}\right)^{\frac{1}{2}} dx = \pi\left(\frac{1}{f}\right) = 2534{\cdot}86 \text{ sec.}$$

The velocity of the marble at any part of its fall inside the earth

given by $$v^2 = 2f\left\{\tfrac{1}{2} + \int_1^s (2-x)\,dx\right\} = f(-2+4s-s^2).$$

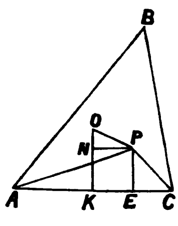
Consequently, the time occupied between the earth's surface and the centre

is given by $$t' = \int_1^2 \frac{dx}{\{f(-2+4x-x^2)\}^{\frac{1}{2}}} = \frac{\pi}{4}\left(\frac{1}{f}\right)^{\frac{1}{2}} = 633{\cdot}715 \text{ sec.}$$

Therefore $T = 4(t+t') = 12674{\cdot}3$ sec. $=$ 3 hr. 31 min. 14·3 sec.

10920. (Professor GENESE, M.A.)—If θ, ϕ, ψ be the angles subtended by the sides of ABC at the point whose areal coordinates are α, β, γ, then $\alpha(\cot A - \cot\theta)$ = similar expressions = half the power of the point with respect to the circum-circle.

Solution by G. HEPPEL, M.A.; R. KNOWLES, B.A.; *and others.*



Selecting the side AC for consideration, to avoid confusion between the letters a and α,

$$PE = \frac{2\beta}{b} = \frac{\beta}{R\sin B}, \qquad OK = R\cos B.$$

Let $OP = \delta$, $KE = NP = c$; then

$$c^2 = \delta^2 - \left(R\cos B - \frac{2\beta}{b}\right)^2$$
$$= \frac{b^2}{4} + 2\beta\cot B - \frac{4\beta^2}{b^2} - (R^2-\delta^2),$$

$$\tan \mathrm{APE} = \frac{b(b+2c)}{4\beta}, \qquad \tan \mathrm{CPE} = \frac{b(b-2c)}{4\beta};$$

$$\therefore \quad 8\beta b^2 \cot\phi = 16\beta^2 - b^4 + 4b^2c^2 = 8\beta b^2 \cot \mathrm{B} - 4b^2(\mathrm{R}^2 - \delta^2);$$

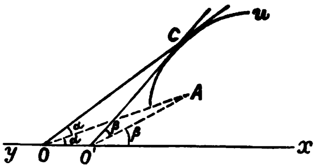
$$\therefore \quad \beta(\cot \mathrm{B} - \cot\phi) = \tfrac{1}{2}(\mathrm{R}^2 - \delta^2).$$

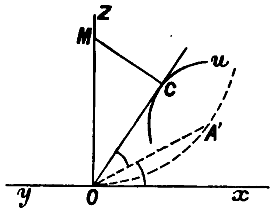
10926. (Professor NEUBERG.)—Une droite roule sur une courbe plane donnée. Déterminer le point où la bissectrice de l'angle que cette droite forme avec un axe fixe, touche son enveloppe.

Solution by Professor DE LONGCHAMPS; H. J. WOODALL; *and others.*

Considérons deux tangentes infiniment voisines à la courbe u; soient 2α, 2β les angles qu'elles forment avec l'axe fixe donné yx. Nous avons

$$\mathrm{OCO}' = 2(\beta - \alpha), \qquad \mathrm{OAO}' = \beta - \alpha.$$





Considérons les cercles circonscrits aux triangles OCO′, OAO′, et soient ρ, ρ' leurs rayons; ils sont donnés par les formules :

$$\frac{\mathrm{OO}'}{\sin 2(\beta - \alpha)} = 2\rho, \quad \frac{\mathrm{OO}'}{\sin(\beta - \alpha)} = 2\rho'; \qquad \therefore \quad \frac{\rho}{\rho'} = \frac{\sin(\beta - \alpha)}{\sin 2(\beta - \alpha)}.$$

Lorsque β tend vers α, on a $\lim. \dfrac{\rho}{\rho'} = \frac{1}{2}$.

D'ailleurs, en menant, en C, la normale à la courbe proposée jusqu'à ce qu'elle rencontre en M la perpendiculaire élevée à yx au point O,

on a $$\mathrm{OM} = \lim. 2\rho = \lim. \rho'.$$

MO est donc le rayon de la circonférence qui passe par O, tangentiellement à OX, et par le point inconnu A′, point situé sur la bissectrice de COX, et qui est la limite du point A. On aura donc A′ en décrivant, de M comme centre, avec MO pour rayon, un arc qui coupe en A′, point cherché, la bissectrice de ZOx.

[The PROPOSER remarks that "le point A est situé sur la bissectrice de l'angle supplémentaire de OCO′; donc il a pour limite l'intersection de la bissectrice de l'angle COx avec la normale à U."]

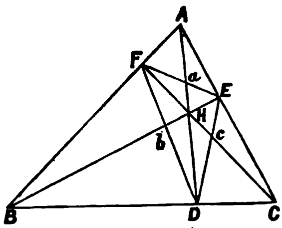
10864. (R. TUCKER, M.A.)—DEF is the pedal triangle of ABC; AD, BE, CF cut its sides in a, b, c, and cointersect in H. Prove that

(1) $\Delta abc = 4\Delta \cos^2 A \cos^2 B \cos^2 C / \{\cos(A-B)\cos(B-C)\cos(C-A)\}$;

(2) perpendiculars from A, B, C, on bc, ca, ab, meet in nine-point centre;

(3) Bc, Cb, ... intersect on AD, ... ; (4) FE, bc, ..., intersect on BC, ...

Solution by Professors RAMASWAMI AIYAR, BEYENS, *and others.*



Put

$$\lambda = \frac{\cos^2 A \cos^2 B \cos^2 C}{\cos(A-B)\cos(B-C)\cos(C-A)}.$$

Now $\quad Aa : aH = AD : HD$;

$$\therefore\ aH = \frac{AH \cdot HD}{AD + HD}$$

$$= 2R \cos A \cos B \cos C / \cos(B-C).$$

$$\therefore\ \Delta abc = \tfrac{1}{2}\Sigma\, bH \cdot cH \sin A$$

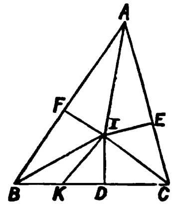
$$= 2R^2 \lambda\, \Sigma \sin A \cos(B-C)$$

$$= 2R^2 \lambda\, \Sigma \sin 2A = 4\Delta\lambda\ (1).$$

Again, if the sides of the triangle DEF be α, β, γ, we have the equations $AD = \alpha - \beta$; $BC = \alpha + \beta$; $bc = \beta + \gamma - \alpha$; $Bc = \beta + 2\gamma - \alpha$; $Cb = \gamma + 2\beta - \alpha$, &c. These equations prove (3) and (4). And it is easy to show that the line joining A $(-1, 1, 1)$ to the N. P. centre of ABC (that is, the centre of the triangle of reference) is perpendicular to bc, that is, $\beta + \gamma - \alpha$, &c.; which proves (2).

10941. (J. J. WALKER, F.R.S.)—The sides of a triangle repelling with a force varying inversely as the cube of the distance (as in Quest. 6120), show that the attractions of the three sides on a particle situate at the centre of the inscribed circle are reducible to three forces perpendicular to the sides and proportional respectively to the angles which they subtend at that point.

Solution by GEORGE HEPPEL, M.A.



Resolving the repulsions of AE, AF along and perpendicular to AI, the latter component is zero, so that the forces are resolved into P, Q, R, in AI, BI, CI, respectively. Resolve P along EI, FI, and similarly with the others. Then total force in DI $= \tfrac{1}{2}(Q \operatorname{cosec} \tfrac{1}{2}B + R \operatorname{cosec} \tfrac{1}{2}C)$.

Now, the repulsion of a particle at K, where

$$DK = x \quad \text{and} \quad \angle IKD = \theta,$$

is $\qquad \mu r^{-3} \sin^3\theta\, dx$, and $x = r\cot\theta$;

$$\therefore\ \text{repulsion in KI} = -\mu r^{-2} \sin\theta\, d\theta.$$

Hence the resolved part in BI $= -\mu r^{-2}\sin\theta\cos(\theta-\frac{1}{2}B)\,d\theta$; and integrating between $\frac{1}{2}B$ and $\frac{1}{2}\pi$, we get $Q = \mu r^{-2}\left[\cos\frac{1}{2}B + (\frac{1}{2}\pi - \frac{1}{2}B)\sin\frac{1}{2}B\right]$; $\therefore$ force in DI $= \frac{1}{2}\mu r^{-2}\left[\cot\frac{1}{2}B + \cot\frac{1}{2}C + \angle BIC\right]$.

Now, $\cot\frac{1}{2}B + \cot\frac{1}{2}C = \dfrac{\sin A}{2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}$; $\therefore$ the three parts of the forces in DI, EI, FI, involving factors such as this, are in equilibrium. There remain therefore only three forces proportional to the angles BIC, CIA, AIB.

10692. (Professor Orchard.)—If a spherical soap-bubble be electrified in such a way that the internal and external air-pressures are equal when the bubble is in equilibrium, how does the tension of the film vary with the electric density?

Solution by Professors Zerr, Bhattacharya, *and others.*

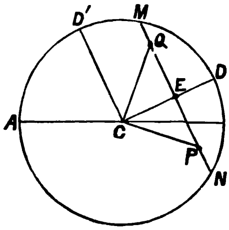
We have (Ex. 9, p. 487, of Minchin's *Statics*, Vol. II.), when the internal and external air pressures are equal,

$$\pi\sigma^2 = \frac{t}{r}; \quad \text{but} \quad V = \frac{Q}{r}, \text{ and } Q = 4\pi r^2\sigma;$$

therefore $$t = \frac{V^2}{16\pi r}; \quad \sigma = \frac{V}{4\pi r}; \quad \text{therefore} \quad \frac{t}{\sigma} = \frac{V}{4} = \frac{Q}{4r}.$$

10933. (The Editor.)—If P, Q be two random points inside a circle whose centre is C, find the average of (1) the perimeter, (2) the area, (3) the sum of the squares on the sides, of the triangle CPQ; also the respective probabilities that, in one such random triangle, the said (4) perimeter, (5) area, (6) sum of squares, will be less than given magnitudes.

Solution by Professor Zerr *and* D. Biddle.



Let P and Q be the two random points in the circle AMDN. Let $AC = r$, $CE = w$, $MP = x$, $PQ = y$, $ME = EN = u$, $\angle D'AC = \theta$. An element of the circle at P is $dw\,dx$; at Q, $y\,d\theta\,dy$. The limits of θ are 0 and $\frac{1}{2}\pi$ and doubled; of w, 0 and r and doubled; of x, 0 and $2u$; of y, 0 and x and doubled.

$u^2 = (r^2 - w^2)$, $\quad CQ = \left\{(u-x+y)^2 + w^2\right\}^{\frac{1}{2}}$,

$CP = \left\{(x-u)^2 + w^2\right\}^{\frac{1}{2}}$, $\quad$ area CPQ is $\frac{1}{2}wy$.

Since the whole number of ways the two

points can be taken in the circle is $\pi^2 r^2$, we have

$$(1)\quad \Delta = \frac{8}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^{2u}\int_0^x \left[y + \left\{w^2+(u-x+y)^2\right\}^{\frac{1}{2}} + \left\{w^2+(x-u)^2\right\}^{\frac{1}{2}}\right] y\,d\theta\,dw\,dx\,dy$$

$$= \frac{4}{3\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^{2u}\Big[2x^3+3x^2\sqrt{w^2+(x-u)^2}+2\left(w^2+u^2\right)^{\frac{3}{2}}$$
$$+3u(x-u)\sqrt{w^2+u^2}+3w^2(x-u)\log\left(u+\sqrt{w^2+u^2}\right)$$
$$-2\left\{w^2+(u-x)^2\right\}^{\frac{3}{2}}+3(x-u)^2\sqrt{w^2+(u-x)^2}$$
$$-3w^2(x-u)\log\left\{u-x+\sqrt{w^2+(u-x)^2}\right\}\Big]\,d\theta\,dw\,dx$$

$$= \frac{2}{3\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r\Big[16\left(r^2-w^2\right)^2+12r^3\left(r^2-w^2\right)^{\frac{1}{2}}+6r\left(r^2-w^2\right)^{\frac{3}{2}}$$
$$-14rw^2\left(r^2-w^2\right)^{\frac{1}{2}}+6w^2\left(r^2-w^2\right)\log\left(r+\sqrt{r^2-w^2}\right)$$
$$-6w^2\left(r^2-w^2\right)\log w-3w^4\log\left(r+\sqrt{r^2-w^2}\right)+3w^4\log w\Big]\,d\theta\,dw$$

$$= \frac{2r}{15\pi^2}\int_0^{\frac{1}{2}\pi}\left(\frac{128}{3}+\frac{273\pi}{16}\right)d\theta = \frac{r}{15\pi}\left(\frac{128}{3}+\frac{273\pi}{16}\right).$$

$$(2)\quad \Delta_1 = \frac{8}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^{2u}\int_0^x \tfrac{1}{2}wy^2\,d\theta\,dw\,dx\,dy = \frac{4}{3\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^{2u} wx^3\,d\theta\,dw\,dx$$

$$= \frac{16}{3\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r \left(r^2-w^2\right)^2 w\,d\theta\,dw = \frac{8r^2}{9\pi^2}\int_0^{\frac{1}{2}\pi} d\theta = \frac{4r^2}{9\pi}\,;$$

$$(3)\quad \Delta_2 = \frac{8}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^{2u}\int_0^x \left[y^2+w^2+(u-x+y)^2+w^2+(x-u)^2\right] y\,d\theta\,dw\,dx\,dy$$

$$= \frac{4}{3\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^{2u}\left[5x^4+6w^2x^2+6u^2x^2-8ux^3\right]d\theta\,dw\,dx$$

$$= \frac{64}{3\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^r\left[\left(r^2-w^2\right)^{\frac{5}{2}}+w^2\left(r^2-w^2\right)^{\frac{3}{2}}\right]d\theta\,dw$$

$$= \frac{4r^2}{4}\int_0^{\frac{1}{2}\pi} d\theta = 2r^2.$$

10868. (Morgan Brierley.)—A sum of money, P, is lent out to interest for seven years, at the end of which time principal and interest, which is to be compound, and reckoned quarterly, amount to 2P. The rate per cent. increasing in the same ratio as the principal, show that it is, for the first and last quarters of the term, £7. 5*s*. 5½*d*. and £14. 6*s*. 8½*d*. per cent. per annum respectively.

Solution by D. Biddle.

Let P = unity, and m_n = what it becomes at the end of the nth quarter. Then $m_{28} = 2$. Moreover, $m_{n-1}(1+m_n x) = m_n$, where x = an

unknown constant. Such being the case, it is easy to go back, finding m_{n-1} in terms of m_n and x. Thus

$$m_{27} = 2/(1+2x), \quad m_{26} = 2/(1+4x),$$

and generally, $$m_n = 2/\{1+2(28-n)x\}.$$

Now $$m_0 = 1 = 2/(1+56x),$$

$$\text{whence } x = \frac{1}{56}; \text{ therefore } m_n = \frac{56}{56-n}, \text{ and } m_n x = \frac{1}{56-n},$$

$$m_1 x = \frac{1}{55} = \text{the rate of interest for the first quarter,}$$

$$m_{28} x = \frac{1}{28} = \text{the rate of interest for the last quarter;}$$

and these respectively are what are stated in the Question.

10884. (C. Leudesdorf, M.A.)—Solve the equation

$$\left\{x - \tfrac{1}{2}a(1-9a^4)\right\}^2 + \left\{x^3 - \frac{1+15a^4}{6a}\right\}^2 = \frac{(1+9a^4)^3}{36a^2},$$

and show that it has four equal roots when $a = \pm(45)^{-\frac{1}{4}}$.

Solution by R. Tucker, M.A.; Prof. Ignacio Beyens, *and others.*

The equation can be put into the form

$$[x - \tfrac{1}{2}a(1-9a^4)]^2 - [\tfrac{1}{2}a(1-9a^4)]^2$$
$$+ \left[x^3 - \frac{1+15a^4}{6a}\right] - \left[\frac{1+15a^4}{6a}\right]^2 = 5a^6 - \frac{a^2}{3},$$

i.e., $$x^6 - \frac{1+15a^4}{3a}x^3 + x^2 - a(1-9a^4)x + \frac{a^2}{3} - 5a^6 = 0.$$

Dividing by $x-a$, we get

$$x^5 + ax^4 + a^2x^3 - x^2\left(\frac{1}{3a} + 4a^3\right) + \left(\frac{2}{3} - 4a^4\right)x + 5a^5 - \frac{a}{3} = 0.$$

Dividing twice by $x-a$, we have, in succession,

$$x^4 + 2ax^3 + 3a^2x^2 - x\left(\frac{1}{3a} + a^3\right) + \frac{1}{3} - 5a^4 = 0,$$

$$x^3 + 3ax^2 + 6a^2x + 5a^3 - \frac{1}{3a} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (a).$$

Hence, that there may be four equal roots, (a) must be satisfied by $x = a$; *i.e.,* $45a^4 = 1$; therefore, &c.

10877. (Professor NEUBERG.)—Les droits qui joignent les sommets d'un triangle ABC à deux points P, Q de son plan rencontrent les côtés opposés en six points d'une même conique. Le point P étant supposé fixe, et la conique étant une hyperbole équilatère, on demand les lieux du point Q et du centre de l'hyperbole.

Solution by Professors CURTIS, NILKANTHA SARKAR, *and others.*

The equation of the conic will be of the form

$$ll'\alpha^2 + mm'\beta^2 + nn'\gamma^2 - (mn' + m'n)\,\beta\gamma - (nl' + n'l)\,\gamma\alpha - (lm' + l'm)\,\alpha\beta = 0$$

(see CASEY, p. 316), $\alpha = 0 \ldots$ being equations of the sides of ABC. The coordinates of P and Q being (l^{-1}, m^{-1}, n^{-1}), and $(l'^{-1}, m'^{-1}, n'^{-1})$. But the condition that $(abcfgh)(\alpha\beta\gamma) = 0$ should be an equilateral hyperbola is

$$a + b + c - 2f\cos A - 2g\cos B - 2h\cos C = 0.$$

We have, also, $\quad ll' + mm' + nn' + 2\,(mn' + m'n)\cos A + \ldots = 0;$

l, m, n being constants, put $\alpha\beta\gamma$ for $1/l'$, $1/m'$, $1/n'$, and

$$l/\alpha + m/\beta + n/\gamma - 2\,(m/\gamma + n/\beta)\cos A - \ldots = 0$$

is a conic passing through A, B, C, and the locus of Q. The locus of the centre of the hyperbola is the nine-point circle of the triangle formed by the three points where the lines from A, B, C through P meet the sides. They are $(0, m^{-1}, n^{-1})$, $(l^{-1}, 0, n^{-1})$, $(l^{-1}, m^{-1}, 0)$.

10156. (R. HOLMES, B.A.)—Solve the differential equation

$$(1 + a^2x^2)^2\,\frac{d^2y}{dx^2} + b^2y = 0.$$

Solution by Rev. J. L. KITCHIN, M.A.; KATE GALE; *and others.*

Assume $ax = \tan t$; then $\dfrac{d^2y}{dt^2} - 2\tan t\,\dfrac{dy}{dt} + \dfrac{b^2}{a^2}\,y = 0$; if $y = Vw$,

and w satisfy $2\dfrac{dw}{dt} - 2\tan tw = 0$, then the equation for V is

$$\frac{d^2V}{dt^2} + IV = 0, \quad \text{where } I = \frac{b^2}{a^2} + 1 + \tan^2 t - \tan^2 t = \frac{a^2 + b^2}{a^2}.$$

Hence we have $$\frac{d^2V}{dt^2} + \frac{a^2 + b^2}{a^2}\,V = 0;$$

therefore $V = A\cos\dfrac{(a^2 + b^2)^{\frac{1}{2}}}{a}\,t + B\sin\dfrac{(a^2 + b^2)^{\frac{1}{2}}}{a}\,t.\quad \dfrac{dw}{dt} - \tan t\,.\,w = 0$

gives $w = \sec t$, no constant being needed. Therefore $y = Vw$

$$= \sec(\tan^{-1} ax)\left\{A\cos\frac{(a^2 + b^2)^{\frac{1}{2}}}{a}\tan^{-1} ax + B\sin\frac{(a^2 + b^2)^{\frac{1}{2}}}{a}\tan^{-1} ax\right\},$$

the complete solution.

10624. (Professor MORLEY.)—Prove that parallels to the sides of a triangle through its Symmedian point meet any cubic of which these sides are asymptotes to six concyclic points.

Solution by Professor ANDERSON, M.A.; H. J. WOODALL; *and others.*

The equations of a cubic whose asymptotes are the sides of the triangle of reference, and of the lines through the Symmedian point parallel to the sides, are $\alpha\beta\gamma - \kappa^2(l\alpha + m\beta + n\gamma) = 0$, $(\alpha - \mu a)(\beta - \mu b)(\gamma - \mu c) = 0$,

where $$\mu = 2\Delta/(a^2+b^2+c^2).$$

Subtracting, we get $$-\kappa^2(l\alpha + m\beta + n\gamma)$$

$$+\mu(a\beta\gamma + b\gamma\alpha + c\alpha\beta) - \mu^2(abc + \beta ac + \gamma ab) + \mu^3 abc = 0,$$

or $$a\beta\gamma + b\gamma\alpha + c\alpha\beta + A\alpha + B\beta + C\gamma = 0,$$

which is a circle. Hence the six points are concyclic.

10873. (Professor MORLEY, M.A.)—There are 4 points in a plane, and each set of 3 is inverted with regard to the fourth: show that the 4 inverse triangles so obtained are similar.

Solution by Professor GENESE, M.A.

If A′B′C′ be the inverse of ABC with respect to D, we have

$$\frac{B'C'}{BC} = \frac{k^2}{BD \cdot DC}, \quad \frac{C'A'}{CA} = \frac{k^2}{DC \cdot DA};$$

therefore $$B'C' : C'A' = BC \cdot DA : CA \cdot DB;$$

so $$C'A' : A'B' = CA \cdot DB : AB \cdot DC.$$

Thus the sides of A′B′C′ are proportional to the rectangles contained by the opposite sides and diagonals of the quadrilateral, &c.

10879. (Professor GENESE, M.A.)—With any point O in the plane of an ellipse as centre, two real circles can be drawn in either of which triangles can be inscribed whose sides touch the ellipse; the radius R of either is given by $(R^2 - OS^2)(R^2 - OH^2) = 4b^2R^2$.

Solution by W. J. GREENSTREET, M.A.; Prof. AIYAR; *and others.*

By SALMON, Art. 132, Ex. 2, we have

$$\Sigma\beta\gamma \sin A = -\sin A \sin B \sin C\, t^2,$$

and $$b^2\Sigma\beta\gamma \sin A = \Delta/R\, \alpha\beta\gamma \text{ (an identity)},$$

therefore $b^2t^2 = -2R\alpha\beta\gamma$, and for the second focus $b^2t'^2 = -2R\alpha'\beta'\gamma'$,

and $$\alpha = \frac{b(c\cos\alpha - a)}{(a^2\sin^2\alpha + b^2\sin^2\alpha)^{\frac{1}{2}}}, \text{ \&c., therefore } t^2t'^2 = 4b^2R^2.$$

10221. (Professor CATALAN.)—Intégrer

$$\frac{(\lambda-1)\cos(\lambda+1)x+(\lambda+1)\cos(\lambda-1)x}{\cos^2 x}\,dx.$$

Solution by D. J. GRIFFITHS; H. J. WOODALL; *and others.*

$$\text{Integral} = \int \frac{2\lambda\cos\lambda x\cos x + 2\sin\lambda x\sin x}{\cos^2 x}\,dx$$

$$= \int \frac{2\lambda\cos\lambda x}{\cos x}\,dx + \int \frac{2\sin\lambda x\sin x}{\cos^2 x}\,dx$$

$$= \frac{2\sin\lambda x}{\cos x} - \int \frac{2\sin\lambda x\sin x}{\cos^2 x}\,dx + \int \frac{2\sin\lambda x\sin x}{\cos^2 x}\,dx \text{ (by parts)} = \frac{2\sin\lambda x}{\cos x}.$$

10831. (Professor HUDSON, M.A.)—If a right circular cylinder be deformed into a regular tetrahedron, prove that the volume of the cylinder is to the volume of the tetrahedron $= 3\sqrt{6} : \pi$.

Solution by H. J. WOODALL; Professor CHAKRIVARTI; *and others.*

To deform cylinder into tetrahedron whose faces have side $= a$, circumference of right section of cylinder $= 2a$, height of cylinder ($=$ height of equilateral triangle) $= (\sqrt{3}/2)\,a$, area of base of cylinder

$$= \pi \times (\text{radius})^2 = \pi \times (2a/2\pi)^2 = a^2/\pi,$$

volume of cylinder $= a^2/\pi \times \sqrt{3}\,.\,a/2 = \sqrt{3}\,.\,a^3/2\pi$,

area of base of tetrahedron $= \sqrt{3}\,.\,a^2/4$, height $= a\sqrt{2/3}$,

volume of tetrahedron $= a^3\sqrt{2}/12$.

$\therefore$ vol. of cylinder : vol. of tetrahedron $= 3\sqrt{6} : \pi$.

8312. (Professor STEGGALL, M.A.)—Find the equation of the motion of a uniform string under the action of gravity in terms of s, ϕ, t; where s, ϕ, t have their usual significations.

Solution by Professor SEBASTIAN SIRCOM, M.A.

Taking u, v the components of the velocity of any point of the string in the direction of s, increasing, and along the normal to the centre of curvature, we shall have $-g\sin\phi$, $-g\cos\phi$ for the components of the force in these directions; and the equations of motion given in ROUTH's *Rigid Dynamics* become

$$(1)\quad \frac{du}{dt} - v\frac{d\phi}{dt} = -g\sin\phi + \frac{dT}{mds},$$

(2) $\dfrac{dv}{dt}+u\dfrac{d\phi}{dt} = -g\cos\phi+\dfrac{\mathrm{T}}{m\rho}$; with the geometrical conditions

$$(3)\quad \frac{du}{ds}=\frac{v}{\rho}, \qquad (4)\quad \frac{d\phi}{dt}=\frac{dv}{ds}+\frac{u}{\rho}.$$

Also, $\rho = \dfrac{\partial s}{\partial\phi}$; then $u=\dfrac{\partial s}{\partial t}$, and, from (3), $v=\dfrac{\partial u}{\partial\phi}=\dfrac{\partial^2 s}{\partial\phi\partial t}$.

Further, $\dfrac{du}{dt}=\dfrac{\partial^2 s}{\partial t^2}+\dfrac{\partial^2 s}{\partial\phi\partial t}\cdot\dfrac{d\phi}{dt}$; $\qquad \dfrac{dv}{dt}=\dfrac{\partial^3 s}{\partial\phi\partial t^2}+\dfrac{\partial^3 s}{\partial\phi^2 dt}\cdot\dfrac{d\phi}{dt}$.

From (4), $\rho\dfrac{d\phi}{dt}=\dfrac{\partial s}{\partial t}+\dfrac{\partial^3 s}{\partial^2\phi\partial t}$; and (1) and (2) become

$$\frac{\partial^2 s}{\partial t^2}\cdot\frac{\partial s}{\partial\phi}+g\sin\phi\frac{\partial s}{\partial\phi}=\frac{d\mathrm{T}}{md\phi},\quad \frac{\partial^3 s}{\partial\phi\partial t^2}\cdot\frac{\partial s}{\partial\phi}+\left(\frac{\partial s}{\partial t}+\frac{\partial^3 s}{\partial\phi^2\partial t}\right)^2+g\cos\phi\frac{\partial s}{\partial\phi}=\frac{\mathrm{T}}{m}.$$

Eliminating T, we have

$$\frac{\partial^2 s}{\partial t^2}\cdot\frac{\partial s}{\partial\phi}-\frac{\partial}{\partial\phi}\left\{\frac{\partial^3 s}{\partial\phi\partial t^2}\cdot\frac{\partial s}{\partial\phi}+\left(\frac{\partial s}{\partial t}+\frac{\partial^3 s}{\partial\phi^2\partial t}\right)^2\right\}+g\left(2\sin\phi\frac{\partial s}{\partial\phi}-\cos\phi\frac{\partial^2 s}{\partial\phi^2}\right)=0,$$

the equation of motion required.

[The Proposer remarks that, so far as he can see, the assumption $u=ds/dt$ is not correct, and involves a superfluous condition; from the four equations u, v, T can be eliminated, and the result is longer than that given.]

10643. (S. Tebay, B.A.)—There are innumerable pairs of right-angled triangles having the same hypothenuse (c), and such that the differences between the hypothenuse and the sides are a square and twice a square. If (α^2, $2\beta^2$) and (α'^2, $2\beta'^2$) be the differences, show (i) that

$$c=(\alpha+\beta)^2+\beta^2=(\alpha'+\beta')^2+\beta'^2;$$

find (2) the sides in integers; and (3) investigate general formulæ for the x^{th} pair of triangles of the species in which $\alpha=\beta'=1$.

Solution by the Proposer; Prof. Mukhopadhyay, M.A.; *and others.*

With regard to the first part of the statement, we have

$$(mn+pq)^2+(mp-nq)^2=(mn-nq)^2+(mp+nq)^2$$
$$=m^2n^2+p^2q^2+m^2p^2+n^2q^2=c.$$

Hence, if c, a, b and c, a', b' be the sides, we have

$$a=(mn+pq)^2-(mp-nq)^2,\qquad b=2(mn+pq)(mp-nq);$$
$$a'=(mp+nq)^2-(mn-pq)^2,\qquad b'=2(mn-pq)(mp+nq);$$
$$\therefore\quad c-a=2(mp-nq)^2,\qquad c-b=\{(mn+pq)-(mp-nq)\}^2;$$
$$c-a'=2(mn-pq)^2,\qquad c-b'=\{(mp+nq)-(mn-pq)\}^2.$$

Again, $(c-\alpha^2)^2+(c-2\beta^2)^2 = c^2$; $\quad\therefore\quad c^2-2(\alpha^2+2\beta^2)c+\alpha^4+4\beta^4 = 0.$

But $$\alpha^4+4\beta^4 = (\alpha^2+2\beta^2+2\alpha\beta)(\alpha^2+2\beta^2-2\alpha\beta).$$

$$\therefore\quad c = (\alpha+\beta)^2+\beta^2, \quad\text{or}\quad c = (\alpha-\beta)^2+\beta^2;$$

so $$c = (\alpha'+\beta')^2+\beta'^2, \text{ or } c = (\alpha'-\beta')^2+\beta'^2;$$

that is, $$c = (\alpha+\beta)^2+\beta^2 = (\alpha'+\beta')^2+\beta'^2.$$

It does not, however, follow that $(\alpha-\beta)^2+\beta^2$ is equal to $(\alpha'-\beta')^2+\beta'^2$, unless β and β' are negative.

To determine the sides, $\quad b = c-2\beta^2$;

$$\therefore\quad a^2 = 4\beta^2(c-\beta^2) = 4m^2\beta^2, \text{ suppose};$$

$$\therefore\quad c = m^2+\beta^2 \quad\text{and}\quad a = 2m\beta.$$

Again, $\quad b' = m^2+\beta^2-2\beta'^2 \quad$ and $\quad a'^2 = 4\beta'^2(m^2+\beta^2-\beta'^2)$.

Let $\quad m^2+\beta^2-\beta'^2 = (m-2k)^2$; $\quad$ then $\quad m = k+(\beta'^2-\beta^2)/4k$.

Let $\quad \beta' = 4ks+\beta$; $\quad$ then $\quad m = k+(\beta+\beta')s = k+2(2ks+\beta)$;

hence the sides are

$$m^2+\beta^2,\quad 2m\beta,\quad m^2-\beta^2,\quad m^2+\beta^2,\quad 2\beta'(m-2k),\quad m^2+\beta^2-2\beta'^2.$$

Take $k = 2$, $s = 1$, $\beta = -1$; then $\beta' = 7$, $m = 8$, $c = 65$. Hence the sides are 65, 16, 63, and 65, 56, 33.

This is a general but implicit solution of the problem, but it affords but little information in particular cases; for instance, when $\alpha = 3$, and $\beta' = 7$. In this case we have

$$a = c-2\beta'^2,\quad b = 2\beta'^2(2c-2\beta'^2);\qquad\therefore\quad c = m^2+\beta'^2.$$

Again, $\quad a' = c-\alpha^2 = m^2+\beta'^2-\alpha^2$; $\qquad\therefore\quad b'^2 = \alpha^2(2m^2+2\beta'^2-\alpha^2)$.

$$\therefore\quad 2m^2+2\beta'^2-\alpha^2 = \text{square}.$$

This expression can be made a square when $2\beta'^2-\alpha^2$ is a square.

Let $\qquad \alpha = 3,\ \beta' = 7$; $\qquad$ then $2m^2+89 = $ a square.

Let $m = \rho+4$; then $2m^2+89 = 2\rho^2+16\rho+121 = (11\pm\rho p/q)^2$, suppose.

$$\therefore\quad \rho = \frac{2q(8q\pm 11p)}{p^2-2q^2}.$$

If p/q be a convergent to $\sqrt{2}$, then $p^2-2q^2 = 1$, and $\rho = 2q(8q\pm 11p)$.

Therefore $$m = 4+2q(8q\pm 11p).$$

The first convergent to $\sqrt{2}$ is $p/q = 1/1$, which makes $m = -34$, $c = 1205$. Thus the sides are 1205, 1196, 147, and 1205, 1107, 476.

The next convergent $p/q = 3/2$, giving 4145, 4136, 273; 4145, 4047, 896.

Let $\alpha = 1$, $\beta' = 1$. Then $a = c-2$, $b^2 = 4(c-1) = 4m^2$;

therefore $$c = m^2+1.$$

Also, $a' = m^2$, and $b'^2 = 2m^2+1 = (1-p/q)^2$;

therefore $$m = 2pq/(p^2-2q^2).$$

If p/q be a convergent to $\sqrt{2}$, then $p^2-2q^2 = 1$, $m = 2pq$, and

$$c = 4p^2q^2+1.$$

Now, $$\sqrt{2} = 1+\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\&c.$$

Consider $\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}$ &c. The successive convergents are

$$\frac{1}{a},\quad \frac{a}{a^2+1},\quad \frac{a^2+1}{a^3+2a},\quad \frac{a^3+2a}{a^4+3a^2+1},\quad \frac{a^4+3a^2+1}{a^5+4a^3+3a},\quad \text{\&c.}$$

If p_x/q_x be the x^{th} convergent, we have

$$p_{x+2} = p_x + ap_{x+1},\quad q_{x+2} = q_x + aq_{x+1}. \qquad \text{Also, } p_{x+1} = q_x.$$

Let $a = 2$; then $q_x + 2q_{x+1} - q_{x+2} = 0$; the complete solution of which is

$$q_x = \sqrt{2}\left\{r^{x+1} + (-1)^x r^{-x-1}\right\}/4;$$

therefore

$$p_x = \sqrt{2}\left\{r^x + (-1)^{x-1} r^{-x}\right\}/4,$$

where r is a root of the equation $r^2 - 2r = 1$, namely $r = 1 + \sqrt{2}$.

Thus, $\quad m = 2(p_x + q_x)q \quad$ and $\quad c = 4(px_2 + q_2x)^2q^2x + 1$.

The convergents $\frac{3}{2}$ and $\frac{7}{5}$ give

145, 441, 17; 4901, 4900, 99; 145, 143, 24; 4901, 4899, 140.

General Formulæ (t any integer).

$c = 25a^2 - 70a\beta + 50\beta^2$,	$c = 25a^2 - 7a\beta + 50\beta^2$,
$a = 24a^2 - 70a\beta + 50\beta^2$,	$a' = 25a^2 - 7a\beta + 48\beta^2$,
$b = a(7a - 10\beta)$,	$b' = 2\beta(5a - 7\beta)$,
$c = 578t^2 - 442at + 85a^2$,	$c = 578t^2 - 442at + 85a^2$,
$a = 578t^2 - 442at + 84a^2$,	$a' = 431t^2 - 538at + 73a^2$,
$b = a(34t - 13a)$,	$b' = 2(23t - 9a)(7t - 2a)$,
$c = 2738t^2 - 2294at + 481a^2$,	$c = 2738t^2 - 2294at + 481a^2$,
$a = 2738t^2 - 2294at + 480a^2$,	$a' = 1151t^2 - 1052at + 238a^2$,
$b = a(74t - 31a)$,	$b' = 2(23t - 9a)(47t - 20a)$.

If we operate first with a^2, we get $c = \frac{1}{2}(m^2 + a^2)$. But

$$c = (a + \beta)^2 + \beta^2;$$

$$\therefore\ \tfrac{1}{2}(m^2 + a^2) = (a + \beta)^2 + \beta^2; \qquad \therefore\ m = a + 2\beta.$$

So

$$m' = a' + 2\beta'.$$

Duplicate Right-Angled Triangles.

65,	63,	16	265,	264,	23	481,	480,	31	785,	783,	56
65,	56,	33	265,	247,	96	481,	360,	319	785,	736,	273
85,	84,	13	305,	273,	136	485,	483,	44	905,	777,	464
85,	77,	38	305,	224,	207	485,	476,	93	905,	663,	616
145,	144,	17	325,	323,	36	505,	377,	336	949,	851,	420
145,	143,	24	325,	253,	204	505,	456,	217	949,	900,	301
185,	153,	104	365,	364,	27	545,	544,	33	1025,	1023,	64
185,	176,	57	365,	357,	76	545,	313,	189	1025,	897,	490
205,	187,	84	377,	345,	152	565,	396,	403	1625,	1624,	57
205,	156,	133	377,	352,	135	565,	493,	276	1625,	1184,	1113
221,	220,	21	425,	416,	87	725,	644,	333	2117,	2115,	92
221,	171,	140	425,	304,	297	725,	627,	364	2117,	2108	195

10549. (J. MacNeill, M.A.)—A borrows from B, on Jan 1st, £500 at 5 per cent. (interest convertible every moment), and B borrows from C on like terms £500 at 10 per cent. on Feb. 1st; find (1) when A's debt will equal C's debt; and (2), if both bills are discounted on March 1st (at 5 or 10 per cent.), how much B has gained or lost.

Solution by H. J. Woodall; A. McMurchy; *and others.*

Amounts of £500 at 5 and 10 per cent. on the nth day are $500e^{n/(7300)}$, $500e^{(n-31)/(3650)}$, nth day counting from January 1st; and when these amounts are equal, $\frac{n}{7300} = \frac{n-31}{3650}$; therefore $n = 62$, *i.e.*, March 4th. On March 1st the difference is in B's favour, and is equal to

$$500\left(e^{\frac{59}{7300}} - e^{\frac{28}{3650}}\right) = 500e^{\frac{28}{3650}}\left(e^{\frac{3}{7300}} - 1\right) = 4/1\tfrac{3}{4} \text{ nearly.}$$

10182. (Prof. Catalan.)—Soit $x = 2^{1-n}(C_{n,1} - 3C_{n,3} + 3^2C_{n,5} - \&c.)$ Si l'on fait $n = 1, 2, 3 \ldots$, les valeurs de x sont $+1$, -1, ou zéro.

Solution by Rev. J. L. Kitchin; R. Knowles, B.A.; *and others.*

$$(1+x)^n = C_{n,0} + C_{n,1}x + C_{n,2}x^2 + \ldots,$$
$$(1-x)^n = C_{n,0} - C_{n,1}x + C_{n,2}x^2 - \ldots;$$

$$\therefore\ (1+x)^n - (1-x)^n = 2x\left\{C_{n,1} + C_{n,3}x^2 + C_{n,5}x^4 + \ldots\right\}: \text{ if } x = (-3)^{\frac{1}{2}},$$

$$\therefore\ \left[1+(-3)^{\frac{1}{2}}\right]^n - \left[1-(-3)^{\frac{1}{2}}\right]^n = 2i\sqrt{3}\left\{C_{n,1} - C_{n,3}3 + 3^2C_{n,5} - \&c.\right\},$$

$$\left\{\frac{1+(-3)^{\frac{1}{2}}}{2}\right\}^n - \left\{\frac{1-(-3)^{\frac{1}{2}}}{2}\right\}^n = i\sqrt{3}\,2^{1-n}\left\{-\right\},$$

$$\left(\cos\tfrac{1}{3}\pi + i\sin\tfrac{1}{3}\pi\right)^n - \left(\cos\tfrac{1}{3}\pi - i\sin\tfrac{1}{3}\pi\right)^n = \left\{-\right\};$$

$$\therefore\ \frac{2\sin\frac{1}{3}(n\pi)}{\sqrt{3}} = 2^{1-n}\left\{C_{n,1} - 3C_{n,3} + 3^2C_{n,5} - \&c.\right\}.$$

The sinister has the values 1, -1, 0, when n is 1, 2, 3, &c.

10586. (J. O'Byrne Croke, M.A.)—If the general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola, prove that, if Δ be the discriminant, and

$$\frac{d\Delta}{da} = A, \quad \frac{d\Delta}{db} = B, \quad \frac{d\Delta}{dg} = 2G, \quad \frac{d\Delta}{df} = 2F,$$

the vertex is
$$x = \left\{(a+b)^2A - a\Delta\right\} / \left\{2(a+b)^2G\right\},$$
$$y = \left\{(a+b)^2B - b\Delta\right\} / \left\{2(a+b)^2F\right\}.$$

Solution by the PROPOSER.

Since the general equation of the parabola may be written

$$(\alpha x+\beta y)^2+2gx+2fy+c=0,$$

and the equation of the principal diameter is easily seen to be

$$\alpha x+\beta y+\frac{\alpha g+\beta f}{\alpha^2+\beta^2}=0,$$

we have the following two equations to determine the vertex, namely,

$$\alpha x+\beta y=-\frac{\alpha g+\beta f}{\alpha^2+\beta^2},\quad 2gx+2fy=-c-\left(\frac{\alpha g+\beta f}{\alpha^2+\beta^2}\right)^2;$$

therefore $$y=\frac{\alpha c(\alpha^2+\beta^2)^2+(\alpha g+\beta f)(\alpha\beta f-g\alpha^2-2g\beta^2)}{2(\beta g-\alpha f)(\alpha^2+\beta^2)},$$

that is, $$y=\frac{(a+b)^2\,\mathrm{B}+\mathrm{G}^2}{2\mathrm{F}(a+b)^2};\ \text{similarly,}\ x=\frac{(a+b)^2\,\mathrm{A}+\mathrm{F}^2}{2\mathrm{G}(a+b)^2}.$$

But $$\mathrm{G}^2=-b\Delta+\mathrm{AC},\quad \mathrm{F}^2=-a\Delta+\mathrm{BC},$$

and for the parabola, $\mathrm{C}=0$; therefore

$$\mathrm{G}^2=-b\Delta,\quad \mathrm{F}^2=-a\Delta.$$

Hence the foregoing values of x and y may be written as in the question.

10552. (J. O'Byrne Croke, M.A.)—If Δ be the discriminant of the general equation of the second degree,

$$\mathrm{C}=d\Delta/dc,\quad 2\mathrm{G}=d\Delta/dg,\quad 2\mathrm{F}=d\Delta/df,\quad \mathrm{R}^2=4h^2+(a-b)^2,$$

and $h\tan^2\theta+(a-b)\tan\theta-h=0$; prove that—the sign of R being chosen so as to render the quantity under the radical sign positive—the coordinates of the real foci of an ellipse are

$$\left.\begin{matrix}x_1\\x_2\end{matrix}\right\}=\frac{\mathrm{G}}{\mathrm{C}}\pm\frac{(-\Delta\mathrm{R})}{\mathrm{C}}\cos\theta,\quad \left.\begin{matrix}y_1\\y_2\end{matrix}\right\}=\frac{\mathrm{F}}{\mathrm{C}}\pm\frac{(-\Delta\mathrm{R})^{\frac{1}{2}}}{\mathrm{C}}\sin\theta.$$

Solution by the PROPOSER.

In an ellipse, if α, β be the coordinates of the centre, and δ the central distance of each of the foci, then θ, the inclination of the transverse axis to the axis of x, being determined from the equation

$$h\tan^2\theta+(a-b)\tan\theta-h=0,$$

we have $$\left.\begin{matrix}x_1\\x_2\end{matrix}\right\}=\alpha\pm\delta\cos\theta,\quad \left.\begin{matrix}y_1\\y_2\end{matrix}\right\}=\beta\pm\delta\sin\theta.$$

But, writing the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

if A, B be respectively the major and minor axes of the ellipse, we have

$$\delta^2 = A^2 - B^2 = \frac{2\Delta}{(h^2-ab)(a+b-R)} - \frac{2\Delta}{(h^2-ab)(a+b+R)}$$

$$= \frac{4\Delta R}{(h^2-ab)\left\{(a+b)^2-R^2\right\}}.$$

And, since $R^2 = 4h^2+(a-b)^2$, $(a+b)^2-R^2 = 4(ab-h^2)$,

$$\delta^2 = \frac{-\Delta R}{(h^2-ab)^2}; \text{ therefore } \delta = \frac{(-\Delta R)^{\frac{1}{2}}}{h^2-ab} = \frac{(-\Delta R)^{\frac{1}{2}}}{C}.$$

Also $\alpha = \dfrac{bg-hf}{h^2-ab} = \dfrac{G}{C}$; $\beta = \dfrac{af-hg}{h^2-ab} = \dfrac{F}{C}$;

hence the stated results follow.

10913. (The EDITOR.)—Solve the equation

$$\frac{2a-b-c}{x+a-b-c} + \frac{2b-c-a}{x+b-c-a} + \frac{2c-a-b}{x+c-a-b} = 4.$$

Solution by R. TUCKER, M.A.; GEORGE HEPPEL, M.A.; *and others.*

Put $a+b+c=\lambda$; then

$$2x\Sigma a(\lambda-3a) + \Sigma(3a-\lambda)(2b-\lambda)(2c-\lambda)$$

$$= 4\left[x^3-\lambda x^2+x\Sigma(2a-\lambda)(2b-\lambda)+(2a-\lambda)(2b-\lambda)(2c-\lambda)\right];$$

i.e., $x^3-\lambda x^2+\Sigma ab\,.\,x-abc=0$, or $(x-a)(x-b)(x-c)=0$,

and $x=a$, or b, or c.

[This equation might readily be solved by inspection; for, if we put $x=a$, the sinister becomes $1+\dfrac{2b-c-a}{b-c}+\dfrac{2c-b-a}{c-b} = 1+3 = 4$,

and $x=a$ is one root. So the others are b, c; the equation being cubic.]

10874. (Professor ZERR.)—If ΩD, ΩE, ΩF be the perpendiculars from a Brocard-point on the sides of a triangle, prove that the area of the Brocard ellipse is $\pi\left\{\frac{1}{2}R\,\Omega D\,.\,\Omega E\,.\,\Omega F\right\}^{\frac{1}{2}}$.

Solution by R. TUCKER, M.A.; W. J. GREENSTREET, M.A.; *and others.*

$$\Omega D = 2R\sin^2\omega\, c/b;\quad \Omega E = 2R\sin^2\omega\, a/c;\quad \Omega F = 2R\sin^2\omega\, b/a.$$

$$a_1 = R\sin\omega;\quad b_1 = 2R\sin^2\omega.$$

(SIMMONS's *Recent Geometry* of the triangle, § 11, § 17.)

Now $\Omega D\,.\,\Omega E\,.\,\Omega F = 8R^3\sin^6\omega$,

whence area of ellipse

$$= \pi a_1 b_1 = \pi\,.\,2R^2\sin^3\omega = \pi\left\{\tfrac{1}{2}R\,\Omega D\,.\,\Omega E\,.\,\Omega F\right\}^{\frac{1}{2}}.$$

2906 & 2997. (Professor SYLVESTER.)—**Prove that (2906) the continued fraction $F \equiv 1 + \frac{1}{2^{-1}+} \frac{1}{3^{-1}+} \frac{1}{4^{-1}} + \ldots$ is equal to $\frac{1}{2}\pi$; and that (2997) the fractions derived therefrom, by cutting off any number of consecutive initial terms, is always less than unity, but approaches indefinitely near to unity as the number of terms cut off is increased. [*Ex. gr.*, $\frac{1}{4^{-1}+} \frac{1}{5^{-1}+} \frac{1}{6^{-1}+} \ldots$ *ad inf.*—which is one of the fractions so obtained by cutting off three of the initial terms—will be less than unity.]**

Solution by W. J. GREENSTREET, M.A.

With the usual notation, p_n, q_n satisfy

$$r_{n+1} = r_n/n + r_{n-1} \quad \ldots\ldots(1),$$

where $\quad r_1 = 1$ and $r_n = p_n$ if $r_0 = 1$, $\quad r_n = q_n$ if $r_0 = 0$.

If $y = c_1 x + \frac{1}{2}c_2 x^2 + \frac{1}{3}c_3 x^3 + \ldots$, and as by (1), $(1-x^2)\frac{dy}{dx} = y + r_1 + r_0 x$,

we get $\quad y = (r_1 - r_0)\left\{\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} - 1\right\} + r_0 \sin^{-1} x \left(\frac{1+x}{1-x}\right) \quad \ldots\ldots(2).$

Now, writing $\quad \frac{2 \,.\, 4 \,.\, 6 \ldots 2n}{1 \,.\, 3 \,.\, 5 \ldots 2n-1} = z_n,$

we get $\quad \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = 1 + x + \frac{x^2+x^3}{z_1} + \frac{x^4+x^5}{z_2} + \ldots,$

$$\sin^{-1} x \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = x + \left(\frac{x^2}{2} + \frac{x^3}{3}\right) z_1 + \left(\frac{x^4}{4} + \frac{x^5}{5}\right) z_2 + \ldots;$$

therefore (2) becomes $\quad r_n = (r_1 - r_0)\frac{n}{z_{\frac{1}{2}n}} + c_0 z_{\frac{1}{2}n}.$

Hence for $r_0 = 1$, $r_0 = 0$, we get $p_n = z_{\frac{1}{2}n}$, $q_n = n/z_{\frac{1}{2}n}$; therefore

$$\frac{p_n}{q_n} = \frac{1}{n}(z_{\frac{1}{2}n})^2 = F_n \text{ (say)} \quad \ldots\ldots(3).$$

And the limit of z_n when n is very great is $(\pi n)^{\frac{1}{2}}$; therefore

$$\text{Lt } F_n = \frac{n\pi}{2n} \quad \text{or} \quad F = \tfrac{1}{2}\pi. \quad (2906.)$$

If Q_n be the n^{th} quotient, we have

$$Q_n = \frac{Fq_{n-1} - p_{n-1}}{p_{n-1} - Fq_n} = \frac{q_{n-1}}{q_n} \cdot \frac{F - F_{n-1}}{F_n - F},$$

and we have to find a suitable expression for F_{n-1}.

Stirling's formula gives

$$(z_n)^2 = n\pi + \tfrac{1}{4}\pi\left(1 + \frac{1}{8n} - \frac{1}{32n^2} + \ldots\right);$$

therefore $\quad F_n = \frac{1}{2}\pi + (-1)^n \frac{\pi}{4n}\left(1 + \frac{(-1)^n}{4n} - \frac{1}{8n^2} + \ldots\right).$

Hence $$Q_n = 1 + \frac{1}{2n} + \frac{3}{8n^2} + \dots \text{ and Limit } Q_n = 1. \quad (2997.)$$

[The second part of Quest. 2997 is solved in Vol. xviii., p. 101.]

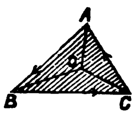
10969. (Professor Curtis, S.J.)—Give a simple method of proving the theorem of moments with regard to a point in a plane.

Solution by G. E. Crawford, B A.

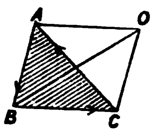
The following plan of treating moments is one I have for some time adopted in teaching. I give no details in sections (1), (2), (3), in which all text-books agree, but (4) and (5) are, I believe, original: at any rate in this connexion. The difficulty of discriminating between inside and outside positions of P (a great stumbling-block to beginners) is removed at a very early stage of the work.

(1, 2, 3) Definition of moment; Convention of sign; Geometrical representation.

(4) *Lemma.*—If AB, BC, CA, the sides of a ΔABC, taken in order, represent forces, the sum of their moments about any external or internal point $= \pm 2\Delta ABC$, the + or − sign corresponding to a +ve or −ve circulation of the arrow-heads which indicate the forces.



(i.)



(ii.)

(i.) Let O be inside; therefore sum of moments

$$= 2\Delta OBC + 2\Delta OBA + 2\Delta OCA$$
$$= 2\Delta ABC.$$

(ii.) Let O be outside; therefore sum of moments

$$= 2\Delta OBC + 2\Delta OBA - 2\Delta OCA = 2\Delta ABC.$$

(5) *Prop.*—The sum of moments of any two forces (AB, AC) about any point (P) = moment of their resultant (AE).

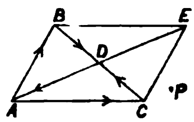
Completing the parallelogram, the diagonals BC, AE bisect each other at D. Then

moments of AB, BD, DA about P $= -2\Delta ABD$;

moments of AC, CD, DA about P $= +2\Delta ACD$;

∴ sum of moments of

AB, AC, BD, CD, 2DA = 0.



But moments of BD, CD together = 0;

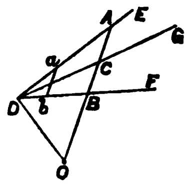
∴ moments of AB, AC, 2DA = 0;

∴ moments of AB, AC together = moment of 2AD = moment of DE.

[The following is Professor CURTIS's "simple method":—Let DE, DF be the directions of the two forces P and Q. Cut off Da, Db proportional to P and Q, and draw OBA parallel to ab. DA and DB may be taken to represent P and Q. DC is half the resultant $= \frac{1}{2}$DG, where DG = resultant.

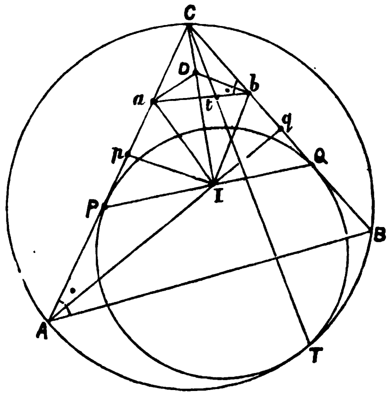
But $\triangle$ADO, BDO, CDO $= \frac{1}{2}$ moment of P, Q, and $\frac{1}{2}$ of resultant round O; and

$$ADO + BDO = 2CD; \quad \therefore \ \&c.]$$



10963. (R. C. J. NIXON, M.A.)—A circle touches the sides CA, CB of a triangle in P, Q, and also touches its circumcircle in T: show that PQ goes through the in-centre, if the contact at T is internal, or through the ex-centre if external.

Solution by Professors NIXON, GENESE, *and* G. HEPPEL, M.A.



Let T be the point of internal contact; and let CI, bisecting angle ACB, meet PQ in I. Invert with respect to C and CI, as centre and radius of inversion.

Circle CAB inverts into ab anti-parallel to AB; so that

$$CA \cdot Ca = CI^2 = CB \cdot Cb,$$

and angle Cba = angle CAB. CT cuts ab in t, the inverse of T.

If p, q are the inverses of P, Q, respectively,

$$CP \cdot Cp = CI^2 = CQ \cdot Cq;$$

therefore, since angles CIP, CIQ are right, so also are angles CpI, CqI.

Therefore Ip, Iq are radii of circle, centre I, touching CA, CB in p, q; and this last circle is obviously the inverse of circle PTQ, and therefore touches ab at t; *i.e.*, I is ex-centre of triangle Cab.

Let D be in-centre of triangle Cab; therefore aIbD is cyclic.

Now, triangles CIA, CaI are similar;

$$\therefore \ \text{angle CAI} = \text{CI}a = \text{D}ba = \tfrac{1}{2}\text{C}ba = \tfrac{1}{2}\text{CAB};$$

therefore I is in centre of triangle CAB.

The proof is similar when the contact is external.

[*Otherwise:*— Take CII′ as axis of x. Then CI $= r \operatorname{cosec} \frac{1}{2}$C;

$CI' = r_c \operatorname{cosec} \frac{1}{2}c$. First circle is $(x-k)^2+y^2 = k^2 \sin^2 \frac{1}{2}C$. PQ is $x = k \cos \frac{1}{2}C$. Circumcircle is

$$x^2+y^2-2Rx\cos\tfrac{1}{2}(A-B)-2Ry\sin\tfrac{1}{2}(A-B) = 0.$$

If circles touch, then either

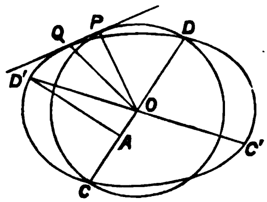
$$k\cos^2\tfrac{1}{2}C = 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B = r\operatorname{cosec}\tfrac{1}{2}C,$$

or

$$k\cos^2\tfrac{1}{2}C = 4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B = r_c\operatorname{cosec}\tfrac{1}{2}C.]$$

10960. (Professor MANNHEIM.)—Sur un diamètre D d'une ellipse donnée on décrit une circonférence de cercle et l'on mène une tangente commune à ces deux courbes. Démontrer que la partie de cette tangente, comprise entre les points de contact, est égale à la projection, sur D, du demi-diamètre qui lui est conjugué.

Solution by Professor GENESE, M.A.; G. HEPPEL, M.A.; *and others.*



The condition that the tangent at $Q(x'y')$ to $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2} = 1$ should also touch the circle $x^2+y^2+2xy\cos\omega = a'^2$ in P is

$$\cos^2\omega-2\cos\omega\frac{x'y'}{b'^2} = \frac{y'^2}{b'^2}\left(1-\frac{a'^2}{b'^2}\right);$$

$$\therefore\quad PQ^2 = x'^2+y'^2+2x'y'\cos\omega-a'^2$$

$$= x'^2+y'^2+b'^2\cos^2\omega-y'^2\left(1-\frac{a'^2}{b'^2}\right)-a'^2 = b'^2\cos^2\omega.$$

[Let the ellipse have semi-axes a, b. Let the given semi-diameter and that at the point of contact be c, h, and the conjugates of these be d, k. Let θ be the angle between c and d. Then we have $cd\sin\theta = ab$; therefore $\cos^2\theta = (c^2d^2-a^2b^2)/c^2d^2$; therefore the square of the projection of d on c is $(c^2d^2-a^2b^2)/c^2$. Now, square of perpendicular from centre on tangent is ab/k, and by hypothesis this $= c$; therefore $k = ab/c$; therefore $h^2 = c^2+d^2-a^2b^2/c^2$; and, finally, the square of the tangent is

$$h^2-c^2 = (c^2d^2-a^2b^2)/c^2.]$$

10667. (The EDITOR.)—If a conic S circumscribe a given triangle ABC, and another conic S′ be drawn touching the sides of the triangle, touching S in a point O, cutting S in the points P, Q; prove that (1) the locus of the point of intersection of PQ and the tangent at O is a nodal cubic; and (2) that this is also the locus of the intersection of the tangent at O with the other two common tangents to S, S′.

Solution by Professors NASH, WOLSTENHOLME, *and others.*

The reciprocal problem is as follows:—If S be a conic inscribed in a triangle, U a conic circumscribed to the triangle, and touching S at O, and if P be the intersection of the other common tangents, and Q, R the other common points: to find the envelope of the lines OP, OQ, OR.

Project two of the vertices of the triangle into the circular points: the conic S becomes a parabola, and the conic U a circle through the focus of the parabola, touching the curve at O.

If am^2, $2am$ be the coordinates of O, the equation of the circle is

$$x^2+y^2-ax\,(l+3m^2)+ay\,(m^3-3m)+3a^2m^2=0.$$

If this circle touches the tangent at the point al^2, $2al$, the value of l is given by $$4l^2+4l\,(m^3-m)+3+6m^2-m^4=0.$$

Hence the coordinates of P are given by $4x=a\,(3+6m^2-m^4)$, $y=a\,(m-m^3)$, and the equation of OP is

$$4mx+(3-m^2)\,y-(6m+2m^3)\,a=0.$$

The envelope of this is

$$3y^4+4y^2\,(x^2+24ax-99a^2)+16a\,(2x-3a)^3=0.$$

This envelope being of the fourth degree and third class, the reciprocal is a nodal cubic.

The points Q, R are given by the equation $l^2+2lm+3=0$, and the equation of either of the lines Oq, Or is of the form

$$4lx-y\,(l^2-3)-2a\,(l^3+3l)=0.$$

Hence the envelope of these lines coincides with the envelope of OP.

The envelope has three cusps at the points $2x=3a$, $y=0$, and $x=15a$, $y^2=-324a^2$.

The bitangent corresponding to the node of the cubic is the line $x=3a$, which meets the curve in the real points $y^2=12a^2$.

[The equations of S, S′ are $bc+ca+ab=0$, $(ax^3)^{\frac{1}{4}}+(by^3)^{\frac{1}{4}}+(cz^3)^{\frac{1}{4}}=0$; $x^2a+y^2b+z^2c=0$ the common tangent; $x^4a+y^4b+z^3c=0$ the common chord; and $(ax+by+cz)^2=4\,(bc+ca+ab)\,(yz+zx+xy)$ the two other common tangents.]

10234. (Professor SCHOUTE.)—Prove that the Hessian covariant $(ab)^2\,a_x^{n-2}\,b_x^{n-2}=0$ of the binary equation $a_x^n=b_x^n=0$ represents, when equalised to 0, every point P, the cubic polar system of which, with reference to $a_x^n=0$, lies equi-anharmonically to the point.

Solution by Professor SEBASTIAN SIRCOM, M.A.

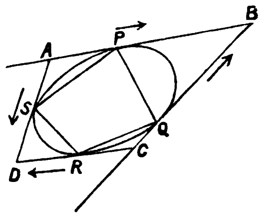
The cubic polar or third emanant of x, y with respect to $a_x^n=0$ is (1) $\left(\xi\dfrac{d}{dx}+y\dfrac{d}{dy}\right)^3 a_{xy}^n=0$; and we need prove the proposition for this cubic alone, since its Hessian is that of the given quantic. Let α, β, γ

be the roots of (1); then $\{(\alpha-\beta)(\gamma-x)\}/\{(\alpha-\gamma)(\beta-x)\}$ is to be equi-anharmonic. A ratio λ is equi-anharmonic if $\lambda^2-\lambda+1=0$; this gives $(\beta-\gamma)^2(\alpha-x)^2+(\gamma-\alpha)^2(\beta-x)^2+(\alpha-\beta)^2(\gamma-x)^2=0$ when reduced to a symmetrical form; but this is $H_n=0$. Hence the result.

10947. (E. M. Langley, M.A.)—Give a statical proof that the locus of the centre of a conic which touches four given straight lines is a straight line.

Solution by Profs. R. Curtis, S.J.; Babel Mullik, M.A.; *and others.*

Let AB, AD, CB, CD represent a system of forces acting in the plane of the paper, P, Q, R, S being the points of contact with an inscribed conic. Force AB≡AP + PB, &c. Resultant of AP and AS passes through the mid-point of SP, and therefore through the centre of the conic. Repeating this for the three other tangent-pairs, we find that the centre must lie on the resultant of the four given forces.



10956. (Professor Morley, M.A.)—On the sides of a triangle a, b, c, draw directly similar triangles x, b, c; a, y, c; a, b, z; then prove that

$$1/(a-x)+1/(b-y)+1/(c-z)=0,$$

the points representing complex quantities in the usual way.

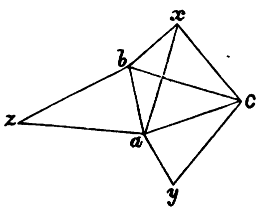
Solution by H. W. Curjel, B.A.; Prof. Genese, M.A.; *and others.*

Let $b-c=r_1(\cos\theta+i\sin\theta)$,

$(c-a)=r_2(\cos\phi+i\sin\phi)$,

$(a-b)=r_3(\cos\psi+i\sin\psi)$;

then the triangles abc, xbc are equal, since they are similar, and have one of their corresponding sides common; therefore $(a-x)=2$ perpendicular from a on $b-c$; therefore length of $a-x=2(2\Delta/r_1)$, where $\Delta=$ area of Δabc.



$\therefore$ $(a-x)=4\Delta/r_1\{\cos(\theta+\frac{1}{2}\pi)+i\sin(\theta+\frac{1}{2}\pi)\}$, since $a-x$ is perpendicular to $b-c$, $=4\Delta i/r_1(\cos\theta+i\sin\theta)$;

$$\therefore\ \Sigma\{1/(a-x)\}=1/4\Delta i\,\Sigma\{r_1/(\cos\theta+i\sin\theta)\}$$
$$=1/4\Delta i\,\Sigma(r_1\cos\theta-ir_1\sin\theta)=0;$$

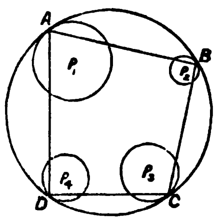
for $\qquad \Sigma r_1 \cos\theta = +\Sigma r \sin\theta = 0;$

for $\qquad \Sigma r_1 \cos\theta + i\Sigma r_1 \sin\theta = \Sigma(b-c) = 0.$

[The geometrical relations are expressed by $\dfrac{x-b}{b-c} = \dfrac{a-y}{y-c} = \dfrac{a-b}{b-z}$; each $= \dfrac{1-x-(y-b)}{y-b} = \dfrac{a-x}{c-z}$; $\therefore \dfrac{1}{y-b} - \dfrac{1}{a-x} = \dfrac{1}{c-z}$, &c.]

10957. (Professor Curtis, S.J., M.A.)—Prove that (1) the condition that four circles S_1, S_2, S_3, S_4 should be touched by another circle is $(12.34 \pm 23.14 \pm 31.24 = 0)i$, where 12 signifies the length of the common tangent to S_1 and S_2, &c.; and (2) if S_1, S_2, S_3 become points 1, 2, 3, and S_1 the circle S, show that 12, 23, 34 become a, b, c (sides of the triangle 1 2 3), and that 14, 24, 34 become t_1, t_2, t_3.

Solution by W. J. Greenstreet, M.A., and Prof. Zerr, M.A.



If the circles, radii $\rho_1, \rho_2, \rho_3, \rho_4$, touch the fifth circle, radius R, in A, B, C, D, respectively,

$$AB^2/(12)^2 = R^2/[(R-\rho_1)(R-\rho_2)] \ldots (\alpha);$$

$$\therefore \quad AB = R(12)/[(R-\rho_1)(R-\rho_2)]^{\frac{1}{2}}, \text{ \&c.}$$

But $AB.CD + BC.AD = AC.BD$;

whence we at once get

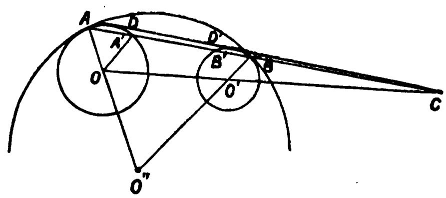
$$(12).(34) + (23).(14) = (13).(24),$$

which, under the given conditions, becomes

$$at_3 + bt_1 = ct_2.$$

If a proof of (α) be needed, it may be obtained thus:—

A′O is parallel to BO″;



$$\therefore \quad \frac{AO''}{OO''} = \frac{AB}{A'B'},$$

i.e., $\dfrac{AB}{A'B} = \dfrac{R}{R-\rho'}$;

or, similarly,

$$\frac{AB}{AB'} = \frac{R}{R-\rho'};$$

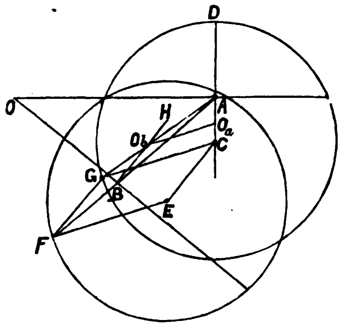
$$\therefore \quad \frac{AB^2}{AB'.A'B} = \frac{R^2}{(R-\rho)(R-\rho')} \quad \text{or} \quad \frac{R^2}{(R-\rho)(R-\rho')} = \frac{AB^2}{DD'^2}.$$

[From this we obtain a proof of Question 10890.]

10845. (Professor Boys, F.R.S.)—Given two lines OA, OB, intersecting at any angle, also points P, Q, one in each line: find (1) two

circular arcs, PR, QR, with radii as $m:n$, to touch the given lines at P and Q, and each other at R; and (2) give also a construction.

Solution by WM. H. MASSEY, M.Inst.C.E., and H. J. WOODALL.



Draw CAD perpendicular to OA; put AC equal to some length; make $AD = n/m \times AC$; draw CE perpendicular to OB from C; and make $CE = AD$. With centres C and E describe circles, radii = CD. Join AB, and produce to meet circle centre E at F; join EF, and draw CG parallel to EF to meet circle C at G. GF is parallel to CE, and therefore perpendicular to OB. Join GA; draw BH perpendicular to OB at B to cut GA at O_b. Draw O_bO_a parallel to GC; then O_a and O_b are the required centres, O_aA and O_bB the radii.

$$CG = EF = CE + CA = GF + AC;$$

hence the circles with centres C and G and radii CA and AD, respectively, would touch each other, and also the line AO and a line through F parallel to OB.

$$FG : O_bB = GA : O_bA = GC : O_bO_a = AC : AO_a \text{ by similar triangles;}$$

$$\therefore \quad O_aA : O_bB = AC : FG,$$

i.e., in the correct proportion. Also,

$$GC = FG + AC; \quad \therefore \quad O_bO_a = O_bB + O_aA;$$

therefore the circles with centres O_a and O_b and radii O_aA and O^bB touch.

[Another solution is given on p. 57 of Vol. LV.]

10967. (Professor GENESE, M.A.)—An ellipse turns about its centre: find (1) the envelope of the chords of intersection with the initial position. Also (2), if the ellipse moves parallel to its major axis, find the envelope of the chords of intersection with the initial position of the axes.

Solution by H. W. CURJEL, B.A.; W. J. GREENSTREET, M.A.; *and others.*

1. Two of the common chords are evidently diameters at right angles; therefore required envelope is that of chords which subtend a right angle at the centre of the ellipse; that is the envelope of

$$x/a\left\{\cos\tfrac{1}{2}(\theta+\phi)\right\} + y/b\left\{\sin\tfrac{1}{2}(\theta+\phi)\right\} = \cos\tfrac{1}{2}(\theta-\phi) \quad \ldots\ldots\ldots(a),$$

where $$a^2 \cos\theta \cos\phi + b^2 \sin\theta \sin\phi = 0,$$
i.e., $$(a^2+b^2)\cos^2\tfrac{1}{2}(\theta-\phi) = a^2\sin^2\tfrac{1}{2}(\theta+\phi) + b^2\cos^2\tfrac{1}{2}(\theta+\phi).$$
Now, perpendicular from centre of ellipse on (α)
$$= \frac{ab\cos\frac{1}{2}(\theta-\phi)}{\left\{a^2\sin^2\frac{1}{2}(\theta+\phi)+b^2\cos^2\frac{1}{2}(\theta+\phi)\right\}^{\frac{1}{2}}} = \frac{ab}{(a^2+b^2)^{\frac{1}{2}}} = \text{constant};$$
therefore envelope is a circle concentric with the ellipse, and its radius
$$= \frac{ab}{(a^2+b^2)^{\frac{1}{2}}}.$$

2. When the ellipse has moved a distance $a\cos\theta$ parallel to the axis of x. The chords of intersection with the axes are
$$\frac{x}{a(\cos\theta \pm 1)} + \frac{y}{b\sin\theta} = 1.$$
Differentiating this, $$\frac{x\sin\theta}{a(\cos\theta+1)^2} - \frac{y\cos\theta}{b\sin^2\theta} = 0.$$
Solving for x and y, $$x = a(\pm 1 + \cos\theta)\cos\theta,$$
$$y = \frac{b\sin^3\theta}{\cos\theta \pm 1} = b\sin\theta(\pm 1 - \cos\theta).$$
This reduces to $$\frac{x}{a} = \pm\left(\frac{y^4}{b^4} + \frac{y^2}{b^2}\right).$$

[If P be a point of intersection of the ellipse in its original and any other position; then OP is equally inclined to the major axes of both ellipses; hence ϕ a common chord which is not a diameter subtends a right angle at the centre, but the join of the extremities of two rectangular diameters touches a fixed circle; thus the envelope is a circle.]

10990. (Rev. C. TAYLOR, D.D.)—If a triangle be circumscribed to a pair of confocal ellipses, prove that the confocal hyperbola through any vertex of the triangle passes through the point of contact of its opposite side.

Solution by W. J. GREENSTREET; Prof. AIYAR, M.A.; *and others.*

If A′, B′, C′ are on $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, and A, B, C are on $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$,

and if A′ be $a\cos\alpha$, $b\sin\alpha$, then A and confocal hyperbolic through A are
$$\frac{a(a^2b^2+2b^2\lambda+\lambda^2)}{a^2b^2-\lambda^2}\cos\alpha,\quad \frac{b(a^2b^2+2a^2\lambda+\lambda^2)}{a^2b^2-\lambda^2}\sin\alpha,$$
$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = a^2-b^2;$$
and this obviously passes through A, since
$$a^2(a^2b^2+2b^2\lambda+\lambda^2)^2 - b^2(a^2b^2+2a^2\lambda+\lambda^2)^2 = (a^2b^2-\lambda^2)^2(a^2-b^2).$$

[*Otherwise*:—If ABC, A′B′C′, PQR be the circum- and in-ellipses, and the triangle formed by tangents at A, B, C; it is readily seen that ABC is the pedal triangle of PQR; hence PA, being perpendicular to QAR, is a tangent to the confocal hyperbola through A; also, PA′ is a tangent to the confocal hyperbola through A′. Again, $\angle$ CBP = PQR,
therefore $\angle$ BPA′ = 90°−PQR = QPA; and $\angle$ BPA′ = QPA,
and PB, PQ are tangents to an ellipse; thus PA′ and PA touch the same confocal ellipse; therefore the confocal hyperbolas through A and A′ are the same.]

NOTE ON QUESTION 10286. *By the* EDITOR.

With reference to our previous note on this Question (p. 115 of Vol. LIV.), we are glad to be able to add the following proof by Mr. W. S. FOSTER, who is of opinion that the theorem is undoubtedly true:—
"In any conic, $(\text{normal})^3 = (\text{semi latus rectum})^2 \times$ radius of curvature. Since the two conics osculate, their radii of curvature at P are equal;
$\therefore\ PQ^3 : PR^3 = L^2 : L'^2$; $\quad \therefore\ PQ : PR = L^{\frac{2}{3}} : L'^{\frac{2}{3}}$ = constant."

3589. (The EDITOR.)—Prove that the sums (S_1, S_2) of the two series

$$\frac{1}{2^4}+\frac{3^2}{2^6 . 3^2}+\frac{3^2 . 5^2}{2^8 . 3^2 . 4^2}+\&c., \quad \frac{3^2}{2^3 . 3 . 4}+\frac{3^2 . 5^2}{2^5 . 3^2 . 4 . 5}+\frac{3^2 . 5^2 . 7^2}{2^7 . 3^2 . 4^2 . 5 . 6}+\&c.$$

are $S_1 = 16/\pi - 5, \quad S_2 = 128/9\pi - 4.$

Solution by Professor SEBASTIAN SIRCOM, M.A.

In the first series, let $s = u_0 + u_1 + \ldots + u_r$; then we have

$$\{2(r-1)+3\}^2 u_{r-1} = (2r+4)^2 u_r,$$

whence

$$\begin{array}{lllll} 9u_0 & & & = 4u_1 & +16 . u_1 + 16u_1, \\ 9u_1 + & 12u_1 & + \quad 4u_1 & = 4 . 2^2u_2 & + 16 . 2u_2 + 16u_2, \\ \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots \\ 9u_{r-1} & + 12(r-1)u_{r-1} & + 4(r-1)^2 u_{r-1} & = 4r^2u_r & + 16 . ru_r + 16u_2. \end{array}$$

Then $9(s-u_r) = 4r^2u_r + 4(u_1 + 2u_2 + \ldots + ru_r) + 12ru_r + 16(s-u_0)$;
therefore $s = 4\{3u_1 + 4u_2 + \ldots + (r+2)u_r\} - 8u_0 + (2r+3)^2 u_r$.
To sum the series in brackets, or $s_2 = w_1 + w_2 + \ldots + w_r$, we have

$$\{2(r-1)+3\}^2 w_{r-1} = 2^2(r+1)(r+2)w_r,$$

that is, $4(r-1)^2 w_{r-1} + 12(r-1)w_{r-1} + 9w_{r-1} = 4r^2w_r + 12rw_r + 8w_r$;
and, proceeding as before,

$$s_2 = (2r+3)^2 w_r - 24w_1 = 2\,\frac{1 . 3^2 . 5^2 \ldots (2r+3)^2}{2^2 . 4^2 \ldots (2r+4)} - 24w_1 = \frac{4}{\pi} - \frac{9}{8},$$

when r is increased without limit. Then, since $u_r = w_r/(r+2)$, $u_\infty = 0$, and the sum required is $4\left(\frac{4}{\pi}-\frac{9}{8}\right)-\frac{1}{2} = \frac{16}{\pi}-5$.

In the second series, $s_1 = v_0+v_1+\dots v_{r-1}$, we shall find

$$9s_1 = (2r+3)^2 v_{r-1}+4\left\{4v_0+5v_1+\dots+(r+3)\,v_{r-1}\right\};$$

and, since $(r+3)\,v_{r-1} = 2^3(r+2)\,u_r$, $9s_1 = 32\left(\frac{4}{\pi}-\frac{9}{8}\right)$, when $r=\infty$, and the sum required is $\frac{128}{9\pi}-4$.

10965. (Professor LAMPE, LL.D.)—Let C be the centre of a rectangular hyperbola, having a contact of the third order at the point (x_1, y_1) with the parabola $y^2 = 2px$. Prove that the equation of the hyperbola is

$$x^2-y^2-2xy\frac{y_1}{p}+2x(p+3x_1)-2y\frac{x_1y_1}{p}+x_1^2 = 0.$$

Solution by W. J. GREENSTREET, M.A.; G. HEPPEL, M.A.; *and others.*

The conic, having contact of the third order with $y^2-2px=0$, is

$$\lambda(y^2-2px)-\{yy_1-p(x+x_1)\}^2.$$

This is a rectangular hyperbola if $\lambda^2 = p^2+y_1^2$; therefore its equation is

$$x^2-y^2-2xy\frac{y_1}{p}+2x(p+3x_1)-2y\frac{x_1y_1}{p}+x_1^2 = 0.$$

The locus of the centre can be easily found, for the centre is given by

$$x+x'+p = 0 \quad\text{and}\quad y = y';$$

therefore the locus of the centre is $y^2+2p(x+p) = 0$, an equal parabola, similarly placed with respect to the directrix of the original parabola.

[Prof. LAMPE remarks that, in sending this question and the two connected therewith, he had especially in view the curious proposition about the centre of the hyperbola. By transformation of the equation, the square of the semiaxis a of the equilateral hyperbola is found to be n^3/p, n being the normal of the parabola in (x_1, y_1), $=(p^2+y_1^2)^{\frac{1}{2}}$. Therefore, putting $R = n^3/p^2$ (= radius of curvature), we have $a^2 = Rp$, the direction of the axis being easily reconstructed from tg $2\alpha = -y_1/p$.

The analogous problem for the ellipse $b^2x^2+a^2y^2=b^2$, instead of the parabola, leads to the equation

$$(a^4y_1^2+b^4x_1^2)(b^2x^2+a^2y^2-a^2b^2) = (a^2+b^2)(b^2xx_1+a^2yy_1-a^2b^2)^2 \;\dots\dots\; \text{(H)}.$$

The centre of this equilateral hyperbola is given by

$$x = \frac{a_1(a^2+b^2)}{a_1^2+y_1^2},\quad y = \frac{y_1(a^2+b^2)}{x_1^2+y_1^2}\;\dots\dots\dots\dots\dots\dots(1, 2).$$

Whence, by inspection, the well-known theorem that the centre lies on the line $x/y = x_1/y_1$, joining the centre of the ellipse to the point of con-

tact. Solving x_1 and y_1 from (1) and (2), and substituting in the equation of the ellipse, the locus of the centre is given by

$$a^2b^2(x^2+y^2)^2 = (a^2+b^2)^2(b^2x^2+a^2y^2),$$

the pedal curve of the ellipse

$$a^2x^2+b^2y^2 = (a^2+b^2)^2$$

for its centre as pole. Finally, the square of the semiaxis a_1 of (H) is obtained in the form

$$a_1^2 = \frac{a^2b^2(a^2+b^2-x_1^2-y_1^2)}{\{x_1^2+y_1^2\}^{\frac{1}{2}}\{b^4x_1^2+a^4y_1^2\}^{\frac{1}{2}}}\Bigg].$$

10977. (W. J. C. Sharp, M.A.)—Prove that

$$F \equiv f(x_1x_2x_3)\dots x_n \equiv \begin{vmatrix} 1+x_1, & 1, & 1, & \dots & 1, & 1, \\ 1, & 1+x_2, & 1, & \dots & 1, & 1, \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1, & 1, & 1, & \dots & 1+x_n, & 1, \\ 1, & 1, & 1, & \dots & 1, & 1, \end{vmatrix} = x_1x_2\dots x_n.$$

Solution by H. W. Curjel, B.A.; G. Heppel, M.A.; *and others.*

Subtracting the last column from the first, we have

$$F = \begin{vmatrix} x_1, & 1, & 1, & \dots & \dots & 1 \\ 0, & 1+x_2, & 1, & \dots & \dots & 1 \\ 0, & 1, & 1+x_3, & \dots & \dots & 1 \\ 0, & 1, & 1, & \dots & \dots & \dots \\ 0, & 1, & 1, & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1, & 1 \\ \dots & \dots & \dots & \dots & 1+x_n, & 1 \\ 0, & 1, & 1, & \dots & 1, & 1 \end{vmatrix} \begin{aligned} &= x_1 f(x_2x_3\dots x_n) \\ &= x_1x_2 f(x_3\dots x_n) \text{ similarly} \\ &= x_1x_2x_3x_4\dots x_{n-1} \\ &\qquad \times \begin{vmatrix} 1+x_n, & 1 \\ 1, & 1 \end{vmatrix} \\ &= x_1x_2x_3x_4\dots x_n. \end{aligned}$$

10379. (Professor de Longchamps.)—Résoudre les équations

$$(2x+b+c)(2x+c+a)(2x+a+b)+(x+a)(x+b)(x+c) = 0 \;\dots\dots(1),$$

$$8(x+a)(x+b)(x+c)+(x+b+c-a)(x+c+a-b)(x+a+b-c) = 0\dots(2),$$

$$8(x+b+c-a)(x+c+a-b)(x+a+b-c)$$
$$+(x+3a-b-c)(x+3b-c-a)(x+3c-a-b) = 0\dots(3);$$

et (4) trouver la *clef* qui permet de former, en nombre indéfini, les équations du même genre.

Solution by Rev. J. L. KITCHIN, M.A.; Prof. MADHAVARAO; *and others.*

1. Write this $(X-a)(X-b)(X-c)+(x+a)(x+b)(x+c)=0$, where $X=2x+a+b+c$; then we have

$X^3-(a+b+c)X^2+X(ab+ac+bc)+x^3+(a+b+c)x^2+(ab+ac+bc)x=0$;

therefore $\qquad X+x=0,$

therefore $\qquad 3x+a+b+c=0, \;\text{ or }\; x=-\frac{1}{3}(a+b+c).$

The other factor, $3x^2+3\Sigma(a)x+\Sigma(bc)=0$, gives the remaining roots.

2. Write this $(Z+2a)(Z+2b)(Z+2c)+(Y-2a)(Y-2b)(Y-2c)=0$,

where $\qquad Z=2x, \quad Y=x+a+b+c;$

therefore $Z^3+2(a+b+c)Z^2+4(ab+ac+bc)Z+Y^3-2(a+b+c)Y^2$
$+4(ab+ac+bc)Y=0;$

therefore $\qquad Z+Y=0, \quad$ *i.e.* $3x+a+b+c=0.$

3. Write this $(P-4a)(P-4b)(P-4c)+(Q+4a)(Q+4b)(Q+4c)=0$,

where $\qquad P=2(x+a+b+c), \quad Q=x-a-b-c;$

then $\qquad P+Q=0, \quad$ whence $3x+a+b+c=0.$

4. Let $\phi(a, b, c, x)$, $\psi(a, b, c, x)$ be any linear functions of a, b, c, x,

e.g., $\qquad lx\pm ma\pm ny\pm rz$, &c.;

then $\qquad (\phi\pm pa)(\phi\pm pb)(\phi\pm pc)+(\psi\mp pa)(\psi\mp pb)(\psi\mp pc)=0$

will give every equation of the kind in the question.

10964. (Professor RAMASWAMI AIYAR.)—Prove that the circle of curvature at P on a parabola cuts the curve in Q. Show that the circle having its centre on the diameter of the parabola through P, and touching the chord PQ at Q, cuts the parabola again at the vertices of an equilateral triangle.

Solution by G. HEPPEL, M.A.; R. KNOWLES, B.A.; *and others.*

Let the parabola be $y^2=lx$; and P be $\left[\frac{k^2}{l}, k\right]$; then, since the circle of curvature passes through three coincident points at P, Q must be $\left[\frac{9k^2}{l}-3k\right]$, and the centre of the second circle must be on the line

$$l^2(y+3k)=2k(lx-9k^2);$$

so that the centre is $\qquad x=(2l^2+9k^2)\div l; \; y=k.$

Transferring the origin to this point, the point Q becomes $[-2l\; -4k]$; and the parabola becomes $(y+k)^2=lx+2l^2+9k^2$, or in polar coordinates $(r\sin\theta+k)^2=lr\cos\theta+2l^2+9k^2$, and the circle is $r^2=4l^2+16k^2=c^2$.

At intersection $\quad c^2\sin^4\theta+4kc\sin^3\theta-\frac{3}{4}c^2\sin^2\theta-2kc\sin\theta+4k^2=0.$

Of this we know one root $c\sin\theta=-4k$, corresponding to the point Q.

The remaining three are given by $\sin^3\theta - \frac{3}{4}\sin\theta + k/c = 0$; therefore, if α, β, γ be the three values of θ,

$$\Sigma \sin\alpha = 0, \quad \Sigma \sin\alpha \sin\beta = -\tfrac{3}{4}.$$

Eliminating γ, we get a quadratic for $\sin\beta$, and the values of β and γ are found to be $\alpha \pm \frac{2}{3}\pi$.

10847. (Professor MARTIN, M.A.)—A tree contains $12n$ apples; $\frac{1}{3}$ of all the apples are rotten, and $\frac{1}{4}$ of all the apples are wormy; find the respective chances that an apple taken at random from the tree will be (1) sound, (2) rotten, (3) wormy, (4) both rotten and wormy.

Solution by Professor G. B. M. ZERR.

As $4n$ apples are rotten, and $3n$ apples are wormy, the tree cannot contain less than $5n$ or more than $8n$ sound apples; hence the respective chances of a sound, a rotten, a wormy, and a wormy and rotten apple are

$$\frac{1}{2}\left(\frac{8n+5n}{12n}\right) = \frac{13}{24}, \quad \frac{1}{2}\left(\frac{n+4n}{12n}\right) = \frac{5}{24},$$
$$\frac{1}{2}\left(\frac{0+3n}{12n}\right) = \frac{1}{8}, \quad \frac{1}{2}\left(\frac{0+3n}{12n}\right) = \frac{1}{8}.$$

10833. (Professor ZERR, M.A.)—Find six positive whole numbers whose sum is a fifth power, and the sum of their fifth powers a fifth power.

Solution by ARTEMAS MARTIN, LL.D.; *the* PROPOSER; *and others.*

In the solution of Quest. 9563 (Vol. L., pp. 74, 75), it is shown that

$$4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5.$$

Let $4m$, $5m$, $6m$, $7m$, $9m$, and $11m$ denote the numbers sought; then we have $(4m)^5 + (5m)^5 + (6m)^5 + (7m)^5 + (9m)^5 + (11m)^5 = (12m)^5.$

This exists for all values of m; thus we must make

$$4m + 5m + 6m + 7m + 9m + 11m = 42m = \text{a fifth power}.$$

Put $m = v^4$, and $42v^4 = v^5$; then $v = 42$, $m = 42^4 = 3111696$, and the numbers required are

$$4 \,.\, 42^4 = 12446784, \quad 5 \,.\, 42^4 = 15558480,$$
$$6 \,.\, 42^4 = 18670176, \quad 7 \,.\, 42^4 = 21781872,$$
$$9 \,.\, 42^4 = 28005264, \quad 11 \,.\, 42^4 = 34228656.$$

Their sum is 42^5, and the sum of their fifth powers is $(12 \,.\, 42^4)^5$.

10611. (Walter Stott.)—Prove that $\dfrac{1}{1+2+3}+\dfrac{1}{4+5+6}+\ldots\equiv$

$$\tfrac{1}{3}\sum_0^n{}_k\,\frac{1}{3k+2}=\tfrac{1}{3}\sum_0^n\int x^{3k+1}\,dx=\tfrac{1}{3}\int_0^1 dx\,\sum_0^n{}_k\,x^{3k+1}=\tfrac{1}{3}\int\frac{x}{1-x}\left\{1-x^{3(n+1)}\right\}dx.$$

Solution by R. Knowles, B.A.

$$S=\tfrac{1}{3}\left(\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{8}+\ldots\text{ to } n \text{ terms}\right)$$

$$=\tfrac{1}{3}\left(\frac{x^2}{2}+\frac{x^5}{2+3}+\frac{x^7}{2+2\,.\,3}+\ldots\frac{x^{3n-1}}{3n-1}\right)\text{ if } x=1;\quad=\tfrac{1}{3}\int_0^1\frac{x\,(1-x^{3n})\,dx}{1-x^3}.$$

10659. (Professor Neuberg.)—Soit I le centre du cercle inscrit au triangle ABC, et soient M, N, P les symétriques de I par rapport à BC, CA, AB. Démontrer que les droites AI, BI, CI concourent au conjugué isogonal d'un certain point de la droite OI (O est le centre du cercle ABC).

Solution by Professor Anderson, M.A.; H. J. Woodall; *and others.*

The trilinear equation of the straight line AM is

$$\beta\,(1+2\cos B)-\gamma\,(1+2\cos C)=0;$$

and, by interchanging the letters, the equations of NB and PC may be written down. These lines intersect in the point whose coordinates are $1/(1+2\cos A)$, $1/(1+2\cos B)$, $1/(1+2\cos C)$, the isogonal conjugate of which is the point whose coordinates are $(1+2\cos A)$, $(1+2\cos B)$, $(1+2\cos C)$; and this latter point lies on the line OI, since its equation is

$$\alpha\,(\cos B-\cos C)+\beta\,(\cos C-\cos A)+\gamma\,(\cos A-\cos B)=0.$$

10822. (J. D. H. Dickson, M.A.)—Find the sum of n terms of the series $u_n\equiv\cos A+2\cos\frac{A}{2}+2^2\cos\frac{A}{2}\cos\frac{A}{2^2}+2^3\cos\frac{A}{2}\cos\frac{A}{2^2}\cos\frac{A}{2^3}+\ldots$

Solution by Prof. Zerr; W. H. Greenstreet; *and others.*

$$u_n=\cos A+2\left\{\cos\frac{A}{2}+2\cos\frac{A}{2}\cos\frac{A}{2^2}+2^2\cos\frac{A}{2}\cos\frac{A}{2^2}\cos\frac{A}{2^3}+\ldots\right\}$$

$$=\cos A+2\left[\frac{\sin A}{2}\left\{\cot\frac{A}{2^n}-\cot\frac{A}{2}\right\}\right]$$

$$=\cos A+\sin A\left\{\cot\frac{A}{2^n}-\cot\frac{A}{2}\right\}=\sin A\cot\frac{A}{2^n}-1.$$

[*Otherwise.*—Sum the series (u_4) to 4 terms; then we have

$$1+u_4 = 2\cos\frac{A}{2}(1+u_3)\ \ldots\ \text{say};\quad 1+u_3 = 2\cos\frac{A}{4}(1+u_2),\ \&\text{c.},$$

until finally $$1+u_1 = 2\cos^2\frac{A}{16};$$

$$\therefore\quad 1+u_4 = 2^4\cos\frac{A}{2}\cos\frac{A}{4}\cos\frac{A}{8}\cos^2\frac{A}{16}$$

$$= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}\,.\,2\sin\frac{A}{4}\cos\frac{A}{4}\,.\,2\sin\frac{A}{8}\cos\frac{A}{8}\,.\,2\sin\frac{A}{16}\cos\frac{A}{16}\,.\cos\frac{A}{16}}{\sin\frac{A}{2}\sin\frac{A}{4}\sin\frac{A}{8}\sin\frac{A}{16}};$$

whence $1+u_4 = \sin A\cot\frac{A}{2^4}$, and generally $u_n = \sin A\cot\frac{A}{2^n}-1.$]

10480. (Professor NIEWENGLOWSKI.)—Démontrer que l'équation

$$(1+p^2+q^2)\,x^2 - \left\{r\,(1+q)^2 + t\,(1+p^2) - 2pqs\right\}x + rt - s^2 = 0$$

a ses deux racines réelles, quels que soient p, q, r, s, t. Trouver la condition pour qu'elle ait une racine double.

Solution by H. J. WOODALL; Prof. NILKANTHA SARKAR; *and others.*

When transformed, the equation is

$$\left\{(1+p^2)\,x-r\right\}\left\{(1+q^2)\,x-t\right\} - (pqx-s)^2 = 0.$$

The values $x = +\infty,\ r/(1+p^2),\ t/(1+q^2),\ -\infty$, give $+, -, -, +$, respectively. Hence the two roots of the equation are real, and lie one on each side of the interval from $r/(1+p^2)$ to $t/(1+q^2)$. The condition for equal roots is $$r : s : t = (1+p^2) : pq : (1+q^2).$$

10627. (Professor ZERR, M.A.)—A tube of uniform cross section, small compared with its length, is bent into the form of a cycloid, its open ends lying at the cusps, and this cycloid is placed with its axis vertical and its vertex downwards. If n fluids are poured in, whose specific gravities are $\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_n$, the arcs occupied by the respective fluids $l_1, l_2, l_3, \ldots l_n$, and no fluid overflowing; and if x is the distance of the free surface of the first fluid from the vertex (measured along the cycloidal arc), prove that

$$4x\,(\sigma_1 l_1 + \sigma_2 l_2 + \sigma_3 l_3 + \ldots + \sigma_n l_n) = \sigma_1 l_1^2 + \sigma_2\,(l_2^2 + 2l_1 l_2) + \sigma_3\,(l_3^2 + 2l_1 l_3 + 2l_2 l_3) + \ldots + \sigma_n\,(l_n^2 + 2l_1 l_n + 2l_2 l_n + \ldots + 2l_{n-1} l_n).$$

Solution by Professors ANDERSON, PROMATHANATH DATA, *and others*.

If Π be the atmospheric pressure, the pressures at the first and second surfaces of separation are $\pi + \left[x^2 - (x-l_1)^2\right] g\sigma_1/8a$,

$$\pi + \left\{\sigma_1\left[x^2-(x-l_1)^2\right] + \sigma_2\left[(x-l_1)^2-(x-l_1-l_2)^2\right]\right\} g/8a.$$

In like manner the pressure at the n^{th} or free surface is

$$\pi + \left\{\sigma_1\left[x^2-(x-l_1)^2\right] + \sigma_2\left[(x-l_1)^2-(x-l_1-l_2)^2\right] + \dots \right.$$
$$\left. + \sigma_n\left[(x-l_1-l_2-l_3-\dots-l_{n-1})^2-(x-l_1-l_2-\dots-l_n)^2\right]\right\} g/8a.$$

But this must be equal to π; hence we have

$$2x(l_1\sigma_1 + l_2\sigma_2 + \dots + l_n\sigma_n)$$
$$= \sigma_1 l_1^2 + \sigma_2(l_2^2 + 2l_1l_2) + \dots + \sigma_n\left[l_n^2 + 2(l_1+l_2+l_3+\dots+l_{n-1})\,l_n\right].$$

9737. (Capitaine DE ROCQUIGNY.)—Soient donnés les deux progressions arithmétiques $1,\ 1+2^m,\ 1+2\,.\,2^m \dots 1+3\,.\,2^m \dots$; $1+2^{m+1},\ 1+2\,.\,2^{m+1},\ 1+3\,.\,2^{m+1}$.. On partage les termes de la seconde en groupes de termes consecutifs, tels que le nombre de termes au (p^e) groupe soit égal au (p^e) terme de la première progression; soit S_p la somme des termes de ce groupe. Démontrer que S_p est un cube parfait. Si l'on intervertit les rôles des deux groupes, la somme S_p' des termes du p^e groupe de la première progression est la somme de deux cubes.

Solution by Professors BEYENS, MUKHOPADHYAY, *and others*.

1. Le nombre des termes qui aura en avant du p^e groupe formé dans la seconde progression sera

$$1+1+2^m+1+2\,.\,2^m+\dots+1+(p-2)\,2^m = \left[1+(p-2)\,2^{m-1}\right](p-1);$$

le premier terme du groupe p^e de la seconde progression est

$$1+2^{m+1}\times\left[1+(p-2)\,2^{m-1}\right](p-1),$$

le dernier terme de ce groupe

$$1+2^{m+1}\left[1+(p-2)\,2_{m-}{}^{1}\right](p-1)+2^{m+1}(p-1)\,2^m,$$

et la somme demandée des termes du p^e groupe de la seconde progression

$$\left\{1+2^{m+1}\left[1+(p-2)\,2^{m-1}\right](p-1)+2^{2m}(p-1)\right\}\left[1+(p-1)\,2^m\right]$$
$$= \left[1+2\,.\,2^m(p-1)+2^{2m}(p-1)^2\right]\left[1+(p-1)\,2^m\right] = \left[1+(p-1)\,2^m\right\}^3.$$

2. Le nombre des termes qui aura en avant du p^e groupe de la première progression est

$$1+1+2^{m+1}+1+2.2^{m+1}+\dots+1+(p-2)\,2^{m+1} = \left[1+(p-2)\,2^m\right](p-1).$$

Le premier terme de ce groupe est

$$1+\left[1+(p-2)\,2^m\right](p-1)\,2^m,$$

et le dernier $\quad 1+\left[1+(p-2)\,2^m\right](p-1)\,2^m+2^m(p-1)\,2^{m+1},$

et la somme est

$$\{1+[1+(p-2)2^m](p-1)2^m+2^{2m}(p-1\}[1+(p-1)2^{m+1}],$$

ou bien

$$= [1+(p-1)2^m+(p-1)(p-2)2^{2m}+2^{2m}(p-1)][1+(p-1)2^{m+1}]$$
$$= [1+2^m(p-1)]^3+[2^m(p-1)]^3.$$

10396. (Maurice d'Ocagne.)—Etant donné un triangle ABC, soit I le point de rencontre de la conjuguée harmonique de la hauteur AH par rapport aux côtés AB et AC, et de la parallèle menée par le milieu M de BC à la bissectrice, intérieure ou extérieure, de l'angle BAC. Démontrer que, si la perpendiculaire menée par I à BC coupe AB en B′ et AC en C′, on a BB′ = CC′.

Solution by R. Knowles; Professor Madhavarao; *and others.*

The equations to AB, AC, AH, AI are, respectively,

$$\gamma = 0, \quad \beta = 0, \quad \cos B\,.\,\beta - \cos C\,.\,\gamma = 0, \quad \cos B\,.\,\beta + \cos C\,.\,\gamma = 0\,;$$

and the equation to MI parallel to $\beta - \gamma = 0$ is

$$a(c-b)\,\alpha - b(b+c)\,\beta + c(b+c)\,\gamma = 0.$$

The equation to the perpendicular from I on BC is

$$(b-c)\cos B\,.\cos C\alpha + b\cos B\,.\,\beta - c\,.\cos C\,.\,\gamma = 0\,;$$

whence we find the coordinates of B′ and C′, and thence

$$BB' = c\,/\,\{a-(b-c)\cos C\} = b\,/\,\{a+(b-c)\cos B\} = CC'.$$

It can be shown that BB′ = CC′ when MI is parallel to the exterior bisector.

10152. (Professor de Longchamps.)—Résoudre l'équation

$$\alpha(x^2-px+q)^2+\beta(x^2+px+q)^2 = x^2.$$

Solution by Rev. J. L. Kitchin; Professor Beyens; *and others.*

$$\alpha(x^2-px+q)^2+\beta(x^2+px+q)^2 = x^2;$$

$$\therefore\quad \alpha\left(x+\frac{q}{x}-p\right)^2+\beta\left(x+\frac{q}{x}+p\right)^2 = 1.$$

Put y for $x+\dfrac{q}{x}$; then we have

$$(\alpha+\beta)\,y^2-2py\,(\alpha-\beta) = 1-(\alpha+\beta)p^2,$$

and $$y = \frac{p(\alpha-\beta)}{\alpha+\beta} \pm \frac{\sqrt{(\alpha+\beta-4\alpha\beta p^2)}}{\alpha+\beta} = 2d, \text{ say.}$$

Then $x + \dfrac{q}{x} = 2d$, and $x = d \pm \sqrt{(d^2-q)}$.

10934. (W. J. C. SHARP, M.A.)—Prove that

$$\begin{vmatrix} 1+x_1, & 1, & 1, & \dots 1 \\ 1, & 1+x_2, & 1, & \dots 1 \\ 1, & 1, & 1+x_3, & \dots 1 \\ \dots & \dots & \dots & \dots \\ 1, & 1, & 1, & 1+x_n \end{vmatrix} = x_1x_2\dots x_n\left\{1+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\dots+\frac{1}{x_n}\right\}.$$

Solution by G. HEPPEL, M.A.; Professor CURTIS, S.J.; *and others.*

Let given determinant be F_n, and let one, the same in all respects except that the last element is 1 in place of $1+x_n$, be f_{n-1}. Then, by subtracting last row but one from last row, we get

$$F_n = x_nF_{n-1} + x_{n-1}f_{n-2}.$$

Also, by a similar subtraction in f_{n-1}, $f_{n-1} = x_{n-1}f_{n-2}$.

But $f_1 = x_1$; $\quad\therefore\; f_{n-1} = x_1x_2x_3\dots x_{n-1}$.

Now, $$F_2 = x_1+x_2+x_1x_2 = x_1x_2\left\{1+\frac{1}{x_1}+\frac{1}{x_2}\right\};$$

$$\therefore\; F_3 = x_3F_2 + x_1x_2 = x_1x_2x_3\left\{1+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right\},$$

and so on.

7420. (By D. EDWARDES, M.A.)—Two billiard balls are moving with equal uniform velocities in the same straight line, B being in front. B impinges directly upon a third ball C at rest. Prove that, if $e < 3-2\sqrt{2}$, there are three collisions between B and C; and find the limit of e, that there may be a third collision between A and B.

Solution by MARGARET T. MEYER; Professor AIYAR; *and others.*

After the impact of two equal balls with velocities u, u', their velocities are, respectively, $\frac{1}{2}u(1+e)+\frac{1}{2}u(1+e)$, $\frac{1}{2}u'(1+e)+\frac{1}{2}u(1+e)$; therefore, taking the initial velocity of A and B as unity,

after first collision of B and C,

$$\text{C's vel.} = \tfrac{1}{2}(1+e), \quad \text{B's vel.} = \tfrac{1}{2}(1-e);$$

after first collision of A and B,

$$\text{B's vel.} = \tfrac{1}{4}(1-e)^2+\tfrac{1}{2}(1+e), \quad \text{A's vel.} = \tfrac{1}{4}(1-e)(1+e)+\tfrac{1}{2}(1-e);$$

after second collision of B and C,

$$\text{C's vel.} = \tfrac{1}{8}(1-e)^2(1+e)+\tfrac{1}{4}(1-e)(1+e)+\tfrac{1}{4}(1+e)^2,$$

$$\text{B's vel.} = \tfrac{1}{8}(1-e)^3+\tfrac{1}{4}(1-e)(1+e)+\tfrac{1}{4}(1+e)^2;$$

after second collision of A and B,

$$\text{B's vel.} = \tfrac{1}{16}(1-e)^4+\tfrac{1}{8}(1-e)^2(1+e)+\tfrac{1}{4}(1-e)(1+e)^2 + \tfrac{1}{4}(1-e)(1+e),$$

$$\text{A's vel.} = \tfrac{1}{16}(1-e)^3(1+e)+\tfrac{1}{8}(1-e)^2(1+e)+\tfrac{1}{4}(1-e)^2 + \tfrac{1}{8}(1-e)(1+e)^2+\tfrac{1}{8}(1+e)^3;$$

after third collision of B and C,

$$\text{B's vel.} = \tfrac{1}{32}(1-e)^5+\tfrac{1}{16}(1-e)^3(1+e)+\tfrac{1}{8}(1-e)^2(1+e)^2 + \tfrac{1}{8}(1-e)^2(1+e)+\tfrac{1}{8}(1-e)(1+e)^2+\tfrac{1}{8}(1+e)^3+\tfrac{1}{16}(1+e)^2(1-e)^2.$$

The conditions for collisions are—

first of A and B, $\quad 1>\tfrac{1}{2}(1-e);$

first of B and C, $\quad \left\{\tfrac{1}{4}(1-e)^2+\tfrac{1}{2}(1+e)\right\}>\tfrac{1}{2}(1+e), \quad \tfrac{1}{4}(1+e)^2>0, \quad e<1;$

second of A and B,

$$\left\{\tfrac{1}{4}(1-e)(1+e)+\tfrac{1}{2}(1-e)\right\}>\left\{\tfrac{1}{8}(1-e)^3+\tfrac{1}{4}(1-e)(1+e)+\tfrac{1}{4}(1+e)^2\right\},$$

$$(1-e)[4-(1-e)^2]>2(1+e)^2;$$

$$(1-e)(1+e)(3-e)>2(1+e)^2, \quad (1-e)(3-e)>2(1+e), \quad (1+e^2-6e)>0;$$

therefore $\quad e<3-2\sqrt{2}, \quad$ **since** $e<1;$

third of B and C is, in addition to this,

$$\left\{\tfrac{1}{16}(1-e)^4+\tfrac{1}{8}(1-e)^2(1+e)+\tfrac{1}{4}(1-e)(1+e)^2+\tfrac{1}{4}(1-e)(1+e)\right\} >\left\{\tfrac{1}{8}(1-e)^2(1+e)+\tfrac{1}{4}(1-e)(1+e)+\tfrac{1}{4}(1+e)^2\right\},$$

$$(1-e)^4>\left\{4(1+e)^2-4(1+e)^2(1-e)\right\}>4e(1+e)^2.$$

The condition for the third collision of A and B is, in addition to this,

$$\left\{\tfrac{1}{16}(1-e)^3(1+e)+\tfrac{1}{8}(1-e)^2(1+e)+\tfrac{1}{4}(1-e)^2+\tfrac{1}{8}(1-e)(1+e)^2+\tfrac{1}{8}(1+e)^3\right\}$$
$$>\left\{\tfrac{1}{32}(1-e)^5+\tfrac{1}{16}(1-e)^3(1+e)+\tfrac{1}{8}(1-e)^2(1+e)^2+\tfrac{1}{8}(1-e)^2(1+e) + \tfrac{1}{8}(1-e)(1+e)^2+\tfrac{1}{8}(1+e)^3+\tfrac{1}{16}(1-e)^2(1+e)^2\right\},$$

$$\tfrac{1}{4}(1-e)^2>\left\{\tfrac{1}{32}(1-e)^5+\tfrac{1}{16}[3(1-e)^2(1+e)^2]\right\}, \quad 8>\left\{(1-e)^3+6(1+e)^2\right\},$$

$$\left\{8-(1-e)^3\right\}>6(1-e)^2, \quad \left\{2^3-(1-e)^3\right\}>6(1+e)^2,$$

$$(1+e)[4+2(1-e)+(1-e)^2]>6(1+e)^2,$$

$$(7-4e+e^2)>(6+6e), \quad (1-10e+e^2)>0,$$

or $\quad e<(5-\sqrt{24})<(5-2\sqrt{6}).$

10809. (Professor MOREL.)—Du foyer F d'une ellipse comme centre, avec le grand axe comme rayon, on décrit une circonférence. Soit A un point quelconque de cette circonférence; du point A, on mène à l'ellipse deux tangentes qui rencontrent la circonférence l'une en B, l'autre en C; démontrer que la ligne BC est tangente à l'ellipse, et qu'elle est perpendiculaire à la droite AF', qui joint le point A au second foyer F'.

Solution by R. KNOWLES, B.A.; Prof. SARKAR, M.A.; *and others.*

Let (h, k) be the coordinates of A;

$(x-ae)^2+y^2=4a^2$, $a^2y_1y+b^2x_1x=a^2b^2$, $a^2y_2y+b^2x_2x=a^2b^2$ ... (1), (2), (3)

the equations to the circle and the tangents AB, AC respectively; substituting from (2), (3) into (1), we obtain the coordinates of B and C,

$$2a\,(ae+x_1)/(a+ex_1)-h, \quad 2ay_1/(a+ex_1)-k, \quad \&c.,$$

and the equation to BC is

$$2\,(ae+h)\,x+2ky = a^2e^2-h^2-k^2 = -2\,(2a^2+aeh-a^2e^2),$$

since $k^2 = 4a^2-(h-ae)^2$, and because

$$(2a^2+aeh-a^2c^2)^2 = a^2\,(ae+h)^2+b^2k^2.$$

BC is a tangent to the ellipse, and is evidently at right angles to AF'.

[Prof. MOREL remarks that "la question est susceptible d'une démonstration géométrique. Joignons le point A au point F, centre du cercle circonscrit au triangle ABC. Puisque AB et AB sont tangentes, l'angle FAB est égal à F'AC. Donc FA étant le rayon du cercle circonscrit, F'A est la hauteur, perpendiculaire à BC. De plus, le cercle FA entrant directeur, donc la symétrique ϕ' de F' par rapport à AC est sur ce cercle; on a ϕ'AC = F'AC. Donc, si AF' rencontre la circonférence en J, on a Cϕ' = CJ; donc, puisque CA est tangente à l'ellipse, CB, perpendiculaire à F'J, est aussi tangente à l'ellipse. Cette seconde démonstration, très simple, indique bien la propriété énoncée, et est une conséquence de ce que les deux foyers sont dans le triangle inscrit, deux points inverses, tels que les a défini le Colonel Mathieu (*Nouvelles Annales*, 1868), et que l'on peut toujours trouver une conique inscrite au triangle, et ayant les deux points inverses comme foyer."]

10857. (G. F. HOWSE, M.A.)—Prove that the locus of the orthocentres of triangles of maximum perimeter inscribed in an ellipse, whose equation is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, is an ellipse of the form $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=k$.

Solution by the PROPOSER.

Let ABC be a triangle of the system; then, taking C, A, B fixed for a maximum value, we must have AC + BC stationary; therefore C must be the point of contact of an ellipse with foci AB drawn to touch the given one. Hence AC, BC make equal angles with the tangent at C,

and consequently, an ellipse, confocal to the given ellipse and touching AC will touch BC also; similarly, it touches AB.

Let its equation be $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$. Then, since triangles can be drawn touching this and inscribed in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we must have $\pm\frac{\alpha}{a}\pm\frac{\beta}{b}=1$; and we know that $\alpha^2-\beta^2=a^2-b^2$. Hence the confocal ellipse is fixed, its axes being given by above equations, and, by SALMON's *Conics*, the locus of the orthocentres will be the conic

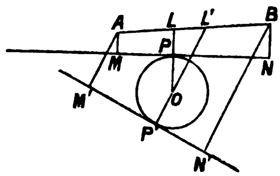
$$x^2+y^2-\alpha^2-\beta^2=\frac{a^2b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right);$$

i.e.,
$$\frac{x^2}{b^2}+\frac{y^2}{a^2}+1=(\alpha^2+\beta^2)\left(\frac{1}{a^2}+\frac{1}{b^2}\right),$$

α, β being roots of the equations $\xi^2-\eta^2=a^2-b^2$, $\xi/a+\eta/b=1$.

10866. (J. D. H. DICKSON, M.A.)—If MPN, M′P′N′ are two tangents to a given circle PQP′; and AM, BN, AM′, BN′ are perpendiculars to them respectively from two fixed points A, B; prove that the tangents are parallel if MP is to PN as M′P′ to P′N′.

Solution by the PROPOSER.



Draw the line AB; if O is the centre of the circle, let OP, OP′ cut AB in L, L′. Then, by similar triangles,

MP : PN = AL : LB;

also, M′P′ : P′N′ :: AL′ : L′B;

whence L and L′ must coincide, and therefore MN is parallel to M′N′.

10629. (Professor GOT.)—Trouver dans l'espace le lieu des points également éclairés par deux points lumineux A et B dont les intensités sont mesurées par les nombres a^2 et b^2. Le point A restant fixe, on propose de déterminer la position du point B sur une circonférence donnée de façon que le lieu géométrique trouvé embrasse un espace maximum.

Solution by Rev. J. L. KITCHIN; Prof. HARKIMA; *and others.*

Let the distance from A to B $= a$; take A as origin of coordinates, AB axis of x; and let P be a point in space whose coordinates are (x, y, z), at which the intensity of light from A and B are equal;

then we have $\quad a^2/(x^2+y^2+z^2) = b^2/\{y^2+z^2+(x-c)^2\}$;

$$\therefore\quad (a^2-b^2)(y^2+z^2)+a^2(x-c)^2-b^2x^2 = 0,$$

or
$$\left(x-\frac{a^2c}{a^2-b^2}\right)^2+y^2+z^2 = \frac{a^2b^2c^2}{(a^2-b^2)^2}.$$

Hence $\dfrac{abc}{a^2-b^2}$ is the radius of the sphere on which, at every point, the intensities are equal.

The greatest space enclosed is evidently that in which c is the greatest possible.

If then we draw from A a line through the centre of the given circle, or sphere, and make this our axis of x, and B at the end of the diameter most remote from A, we get the same equation as before, with radius of sphere, of equal intensities, the greatest possible.

Hence the position of B is determined.

10909. (J. J. Walker, F.R.S.)—If the line which is the locus of the equation $lx+my+nz=0$ meet the sides BC, CA, AB of the triangle of reference in the points A′, B′, C′, and the bisectors of the angles A, B, C in the points D, E, F respectively; prove that

$$m:n = \frac{C'A'}{C'D}:\frac{B'A'}{B'D},\quad n:l = \frac{A'B'}{A'E}:\frac{C'B'}{C'E},\quad l:m = \frac{B'C'}{B'F}:\frac{A'C'}{A'F}.$$

Solution by Professor Gophar; R. Knowles, B.A.; *and others.*

The coordinates of A′, B′, C′, D are (omitting 2Δ) respectively

$$0,\ -n/(cm-bn),\ m/(cm-bn);\quad -n/(cl-an),\ 0,\ l/(cl-an);$$
$$-m(bl-am),\ l/(bl-am),\ 0;$$
$$-(m+n)/(bl+cl-am-an),\quad l/(bl+cl-am-an),\quad l/(bl+cl-am-an).$$

Putting $x = \{a\cos A\,(cm-bn)^2+b\cos B\,(an-cl)^2+c\cos C\,(bl-am)^2\}^{\frac{1}{2}}$,

we get
$$\begin{aligned} A'C' &= (abc)^{\frac{1}{2}}mx/(cm-bn)(bl-am),\\ C'D &= (abc)^{\frac{1}{2}}lx/(bl-am)(bl+cl-am-an),\\ B'A' &= (abc)^{\frac{1}{2}}nx/(cm-bn)(cl-an),\\ B'D &= (abc)^{\frac{1}{2}}lx/(cl-an)(bl+cl-am-an);\end{aligned}$$

therefore $\quad C'A'/C'D : B'A'/B'D = m:n.$

From this result the other ratios can be deduced.

10900. (R. Chartres.) — Show that Delambre's analogies may be immediately derived from Napier's, and give an easy method of remembering them.

Solution by W. J. GREENSTREET, M.A.; Prof. AIYAR; *and others.*

If each of NAPIER's Analogies be added to or subtracted from unity, we get DELAMBRE's Analogies.

Remember $\sin\frac{1}{2}(A+B) / \cos\frac{1}{2}C = \cos\frac{1}{2}(a-b) / \cos\frac{1}{2}c$. On L.H. change B into $-B$, and on R.H. write sin for cos.

10664. (Professor TARRY.) — On donne un point P, une droite A'B', et une conique Σ. Une transversale, tournant autour du point P, coupe la droite en un point Q, et la conique en deux points R, R'. Si l'on prend les points doubles de l'involution déterminée sur la transversale par les deux couples de points P, Q et R, R', le lieu de ces points doubles est une conique Σ'.

Solution by W. J. GREENSTREET, M.A.; FANNIE H. JACKSON; *and others.*

Take the point P for origin, and let the equation of the given A'B' line be $x = \mu$, and that of the conic $(a, b, c, f, g)(x, y, 1)^2 = 0$.

Measuring distances r, r' from the origin we must have for any pair of conjugate points $\quad Arr' + H(r+r') + 1 = 0.$

Hence, using polar coordinates for the line and conic, we see that

$$H\mu\sec\theta + 1 = 0,$$

and $\quad Ac - 2H(g\cos\theta + f\sin\theta) + a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta = 0,$

from which we easily determine A and H. The locus of the double points is

$$Ar^2 + 2Hr + 1 = 0,$$

or $\quad r^2\left\{2\cos\theta(g\cos\theta + f\sin\theta) + \mu(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)\right\} + 2rc\cos\theta - \mu c = 0,$

or $$(a\mu + 2g)x^2 + 2(h\mu + f)xy + \mu by^2 + 2cx - \mu c = 0.$$

10783. (R. CHARTRES, M.A.)—Show that R cannot be less than $2r$.

Solution by Professors ZERR, AIYAR; *and others.*

$(R^2 - 2Rr)^{\frac{1}{2}} = 0$, or $R = 2r$, in case of an equilateral triangle. In case of any other triangle, the distance between the circumcentre and the in-centre is a positive quantity, hence R cannot be less than $2r$.

[The theorem follows immediately from the fact that the distance between the centres of in-circle and circum-circle $= (R^2 - 2Rr)^{\frac{1}{2}}$, and is connected with Quest. 9406.]

APPENDIX I.

UNSOLVED QUESTIONS.

4969. (R. A. Roberts, M.A.)—If from any point of a Cassinian oval tangents be drawn to the curve, prove that their six points of contact lie on a conic.

4979. (Professor Hudson, M.A.)—If a conical cup be filled with liquid, prove that the mean pressure at a point in the volume of the liquid is to the mean pressure at a point in the surface of the cup as 4 : 3.

4984. (Professor Evans, M.A.)—Find the area of the maximum ellipse that can be inscribed in the quadrant of a given circle.

4992. (The Rev. A. F. Torry, M.A.)—Parallel rays fall on one side of a refracting sphere ($\mu = \frac{2}{3}\sqrt{3}$), and form a caustic surface within the sphere. Find the length of that portion of the arc of its generating curve which lies within the sphere.

5000. (Professor Hudson, M.A.) — Find the equation of a curve symmetrical about an axis, such that, when it is immersed in fluid with its highest point at half the depth of its lowest, the centre of pressure may bisect the axis.

5015. (Professor Tanner, M.A.)—Prove that the value common to

$$\int \frac{1}{(x+y)^2} \cdot \int \frac{1}{(x+y)^2} \cdots\cdots \int (x+y)^{2n} \phi(x) \, . \, dx^{n+1},$$

$$\int \frac{1}{(x+y)^2} \cdot \int \frac{1}{(x+y)^2} \cdots\cdots \int (x+y)^{2n} \psi(y) \, . \, dy^{n+1},$$

is of the form $\left\{ (x+y)^2 \dfrac{d}{dx} \right\}^n \dfrac{\chi(x)}{(x+y)^{2n}} + \left\{ (x+y)^2 \dfrac{d}{dy} \right\}^n \dfrac{\theta(y)}{(x+y)^{2n}}$.

5031. (Professor Burnside, M.A.) — If R be the resultant of three conics given by their general equations, prove that

$$R \equiv 6^4 (2\Omega + \Gamma)^2 - 64 \left\{ 108 (\Omega^2 + \Gamma\Omega) - 27\Upsilon \right\},$$

where
$$6\Omega_{11.22.33} = A(B_1C_2 + B_2C_1 - 2F_1F_2) + B(C_1A_2 + C_2A_1 - 2G_1G_2) + \ldots,$$
$$6\omega_{1.2.3} = a(b_1c_2 + b_2c_1 - 2f_1f_2) + b(c_1a_2 + c_2a_1 - 2g_1g_2) + \ldots,$$
$$\Gamma = \omega_{123} - \omega_{122}\omega_{133} - \omega_{233}\omega_{211} - \omega_{311}\omega_{322},$$
$$\begin{aligned} \Upsilon = {} & \Delta_1\Pi_1 + \Delta_2\Pi_2 + \Delta_3\Pi_3 + 2\Delta_1\Delta_2\Delta_3\omega \\ & + 3\omega(\omega_{112}\omega_{223}\omega_{331} + \omega_{113}\omega_{221}\omega_{332}) \\ & - 3(\Delta_2\Delta_3\omega_{112}\omega_{113} + \Delta_3\Delta_1\omega_{221}\omega_{222} + \Delta_1\Delta_2\omega_{331}\omega_{332}) \\ & - 3(\omega_{112}\omega_{113}\omega_{223}\omega_{332} + \omega_{223}\omega_{221}\omega_{331}\omega_{113} + \omega_{331}\omega_{332}\omega_{221}\omega_{112}), \end{aligned}$$
$$\Pi = \begin{vmatrix} \omega_{221} & \omega_{332} & \omega_{331} \\ \Delta_2 & \omega_{223} & \omega_{332} \\ \omega_{223} & \omega_{332} & \Delta_3 \end{vmatrix}.$$

If, in place of considering the three conics U, V, W, we had considered the system U, $V - \frac{\omega_{112}}{\Delta_1}U$, $W - \frac{\omega_{113}}{\Delta_1}U$, which is allowable in calculating a combinant, we get a simple value for Υ, viz. $\Upsilon = \Delta_1 \Pi'$. [Here ω and Δ_1, Δ_2, Δ_3 are written for ω_{123} and ω_{111}, ω_{222}, ω_{333} indifferently.]

5034. (The Rev. W. A. Whitworth, M.A.)—The centre of force in an elliptic orbit is a point in the major axis whose distance from the centre is $a\gamma \sec\gamma$. If two particles describe the orbit, so that one is always half a revolution behind the other; prove that (1) their least distance apart will be $2b\cos\gamma$; and, when they are at this distance, (2) the angular velocity of the straight line joining them is to its mean angular velocity as $a : b(1+\gamma)$.

5035. (A. Martin, LL.D.)—A wooden sphere, of radius r feet, and specific gravity one-n^{th} of water, has a string 6 feet long attached to it, and also to the bottom of a river running with a uniform velocity v; find its position of equilibrium.

5061. (J. Griffiths, M.A.)—If the invariants of two conics, represented by the equations

$$x^2+y^2+z^2 = (lx+my+nz)^2,\quad x^2+y^2+z^2 = (l'x+m'y+n'z)^2,$$

are connected by the relation

$$\frac{1-(ll'+mm'+nn')}{(1-l^2-m^2-n^2)^{\frac{1}{2}}(1-l'^2-m'^2-n'^2)^{\frac{1}{2}}} = 1,$$

it is shown in Dr. SALMON'S *Conics*, 5th ed., that the conics *touch* each other. Find the geometric meaning of the more general relation,

$$\frac{1-(ll'+mm'+nn')}{(1-l^2-m^2-n^2)^{\frac{1}{2}}(1-l'^2-m'^2-n'^2)^{\frac{1}{2}}} = \cos\theta.$$

5065. (A. Martin, LL.D.)—Four spheres, of radii a, b, c, d, touch each other externally, each touching the other three; and four spheres are described in the space enclosed by them, each touching the other three and three of the given spheres. Find the radii of the inscribed spheres.

5096. (The Rev. A. F. Torry, M.A.)—A convex lens is held so that the distance between a bright point and its image is the least possible; two other lenses are then introduced, one half-way between the first lens and the luminous point, the other half-way between the first lens and the image of the point. If the position of the image remains unaltered, the sum of the focal lengths of the three lenses will be zero.

5097. (L. W. Jones, M.A.)—Prove that the director circle of a central conic cuts orthogonally all circles with respect to which circumscribing triangles of the conic are self-conjugate.

5098. (S. Tebay, B.A.)—A number of bricks are placed at equal distances in a straight line; the first falls from a position of equilibrium against the second, which falls against the third, and so on; find the motion of the x^{th} brick, neglecting friction.

5100. (E. B. Elliott, M.A.)—AOA′ and BOB′ being the principal axes of an ellipse, another ellipse is drawn having OA for one axis and the other equal to BB′. Show that, if PQN be any common ordinate, parallel to BB′, of the two, and if on it P′ and Q′ be taken, such that $P'N = F(PN)$ and $Q'N = F(QN)$, then (1) the area contained between the locus of P′, the axis BB′, and the tangent at A, is equal to that between the locus of Q′ and the same lines; (2) if the figure be made to revolve round OA, the volumes generated by these areas shall be equal; and (3) the radii of gyration about OA of the two areas shall be equal, and also those of the two volumes.

5104. (John L. McKenzie.)—In Question 4869, suppose the cubic to be circular; draw through A a line parallel to the real asymptote, and meeting the cubic in M; let Q2 cut the curve in N, and MN in O. Prove that O is on the circumference of the circle of curvature at P.

5106. (J. F. Moulton, M A.)—A ray of light is refracted through a prism in a principal plane. Show that, if the dispersion of two neighbouring colours be a minimum, $\dfrac{\sin(3\phi'-2i)}{\sin\phi'} = 1 - \dfrac{2}{\mu^2}$.

5110. (Professor Crofton, F.R.S.)—A straight uniform bar, weight 50 lbs., is jointed at the centre; and it is found that when placed resting against a wall, with one end on the ground, that end can just be moved out 4 feet from the wall without the bar bending. Determine the stiffness of the joint, as measured by the bending moment that it is able to resist.

5132. (Professor Crofton, F.R.S.)—From any point P within a triangle lines are drawn parallel to the two sides, meeting the base on D, E. Show (1) [as may be done without integration] that the mean value of the triangle PDE is one-sixth of the former triangle; and find (2) the mean value of the segment DE.

5135. (Professor Tanner, M.A.)—A curve, initially in the plane of (x, z), remains parallel to itself while one point on it moves along a curve in the plane of (y, z); find (1) the differential equation of the surface generated; and (2) what the equation becomes when the two planes are those of (x, z), (x, y).

5136. (Professor Hendricks, M.A.)—If four bricks are placed on each other at random, with their longest axes horizontal and in the same vertical plane, show that the probability that the pile will stand is $\frac{209}{2304}$.

5195. (Professor Malet, F.R.S.)—If

$$X \equiv x^4 + px^3 + qx^2 + rx + S, \quad \text{and} \quad U \equiv 36\left(\frac{dX}{dx}\right)^2 - X\left(\frac{d^3X}{dx^3}\right)^2,$$

prove that the expression $$\int \frac{\dfrac{du}{dx}}{uX^{\frac{1}{4}}}\,dx$$

admits of integration in finite terms, and determine the result.

5196. (The Rev. T. P. Kirkman, M.A., F.R.S.)—Among the tens of thousands of 12-acral 12-edra there are $N > 20 < 30$, on which every face and every summit is asymmetrical, while the edges on each form 11 pairs ee', e diametrically opposite to e', such that the configurations completed by e and e', and read by opposite eyes, are one the reflected image of the other. Required the number N.

5225. (The late T. Cotterill, M.A.) — If the sides ab, $a'b'$ of two triangles abc, $a'b'c'$ meet in z, and a transversal cut ca, cb, $c'a'$, $c'b'$ respectively in $\alpha\beta$, $\alpha'\beta'$, prove that the two conics $zab'\alpha\beta'$, $za'b\alpha'\beta$ cut again in three points on the cubic locus of points at which the three pairs of points (aa'), (bb'), (cc') subtend angles in involution.

5235. (The Rev. W. A. Whitworth, M.A.) — Prove that the total number of signals which can be made with n different flags on s different masts is one less than the coefficient of x^n in the expansion of $\underline{|n}.\epsilon^x(1-x)^{-s}$.

5236. (R. A. Roberts, B.A.)—Prove that the circumscribing circle of a triangle, inscribed to a conic and circumscribed to a circle, touches two fixed circles having double contact with the conic.

5238. (A. Martin, LL.D.)—Find the mean distance between two points taken at random within a given right cone.

5240. (J. Richards.)—A, B, C are given points; required the position of the straight line ADE so that the quadrilateral BCED shall be given or a maximum.

5241. (F. C. Wace, M.A.)—To one end of a string which passes over a fixed pulley is fastened a weight nW; to points $A_1, A_2, \ldots A_n$, in the string, distant a apart, are fastened n weights W; the latter are placed close together on a horizontal plane, and motion is allowed to take place. Find the velocity of the system when the last weight W is raised from the plane.

5243. (J. L. McKenzie.)—A line drawn from the centre of an ellipse meets the curve in Q and a fixed tangent in R; and a point P is taken on the line so that $QP : PR = \lambda : \mu$, where $\lambda + \mu = 1$. Show (1) that the locus of P is the quartic

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}-\mu\right)^2 = \lambda^2\left(\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}\right),$$

where ϕ is the excentric angle of the point of contact of the given tangent; (2) investigate the nature of the infinite points on this quartic; also show (3) that the envelope of the quartic, when the given tangent varies, consists of the given ellipse and another similar and concentric ellipse; but if the tangent remain fixed, and the ratio $\lambda : \mu$ vary, the envelope is the fixed tangent, and a line parallel to it through the centre of the ellipse.

5256. (E. B. Elliott, M.A.)—All real values of a, b, c, f, g, h being equally likely, show that the odds are 31 : 1 in favour of the central quadric represented by the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 1$$

being an hyperboloid.

5257. (C. Leudesdorf, M.A.)—Solve, when possible, the simultaneous differential equations

$$\begin{cases} \dfrac{dx}{dt} = a\,x^2 + b\,y^2 + c + 2f\,y + 2g\,x + 2h\,xy, \\ \dfrac{dy}{dt} = a'x^2 + b'y^2 + c' + 2f'y + 2g'x + 2h'xy. \end{cases}$$

5258. (The late T. Cotterill, M.A.)—If a variable conic through four given points a, b, c, d, cut two given lines through d in p and q, prove that the six centres of perspective of the pairs of points p, q, and the three pairs of points taken from the points a, b, c, describe straight lines.

5265. (R. Battle, M.A.)—In a given lemniscate or involute of a circle place a chord inclined to the axis at a given angle, and such that the lines joining its extremities with a given point shall have a given ratio or contain a given angle.

5267. (J. F. Moulton, M.A.)—If θ_1, θ_2 be the angles which the principal normal makes with the axes of x and y, and ϕ_3 the angle that the tangent makes with that of z, then the curves given by $\cos\theta_1 : \cos\theta_2 = x : y$ and $\cos\phi_3 = \dfrac{a^2}{x^2}$ lie on the surfaces $ky = x(z+l)$, the other equation being

$$z = \int \frac{dr}{(r^4 - a^4)^{\frac{1}{2}}}.$$

5269. (R. Levett, M.A.)—If $\dfrac{p}{q}$ and $\dfrac{p'}{q'}$ are two consecutive convergents to a continued fraction x, and s is less than q', prove that

$$\frac{p}{q} \sim x \text{ is less than } \frac{s}{q}\left(\frac{r}{s} \sim x\right).$$

5276. (Professor Evans, M.A.)—ABC is a plane triangle, and OA, OB, OC are lines making equal angles with one another. Find the least integral values of BC, CA, AB that will make OA, OB, OC integral.

5283. (The late T. Cotterill, M.A.)—In a plane take n points and connect them by lines so as to form a polygon of n sides. The polars of the points to a conic form a fresh polygon with n sides corresponding to the n points. Show that the envelope curve determined by the $\frac{1}{2}(n-3)$ lines in the first figure connecting points not already joined, is the reciprocal polar of the locus curve determined by the $\frac{1}{2}n(n-3)$ intersections of the non-consecutive sides of the second figure.

5287. (The Rev. T. P. Kirkman, M.A., F.R.S.)—Of the 176 edges on the 8-acral 8-edra which are the intersections each of a triangle and a quadrilateral, one is taken at random. Required the chance that the number of possible triangular sections, m, a, b, of the solid through m, the mid-point of the edge, and two summits a, b, shall be one and one only.

5296. (L. W. Jones, B.A.)—If from the points of intersection of the sides of a triangle with the bisectors of the opposite angles tangents are drawn to the inscribed circle, and their points of contact joined to the middle points of the corresponding sides of the triangle, show that the three joining lines intersect in a point on the inscribed circle.

5298. (R. A. Roberts, M.A.)—Given five points on a bicircular quartic, prove that the double foci are conjugate with respect to a fixed equilateral hyperbola.

5300. (R. Tucker, M.A.)—Particles are projected with a constant velocity from the same point; through the focus a straight line is drawn parallel to the direction of projection, cutting the path in P, Q. Prove that the locus of P, Q is a sphere with radius equal to twice focal distance of point of projection, and tangential to the directrix plane.

5301. (Rev. E. Hill, M.A.)—Certain persons have imagined the existence of a subterranean connexion between the waters of the Dead Sea and the Mediterranean. Although the difference of their levels is 1300 feet, yet since the ratio of their densities is 1·24, it is possible that such a passage may exist. But find its necessary depth.

5302. (S. Tebay, B.A.)—Given the velocity of wind blowing directly on a wedge-shaped hill; find where, on the other side of the hill, smoke will be driven down chimneys with the greatest possible force.

5321. (Professor Hudson, M.A.)—If the path of a ray cut at a constant angle α the surfaces of equal density in a variable medium, prove that $\mu = \mu_0 \epsilon^{\phi \tan \alpha}$, where ϕ is the inclination of the path to a fixed line.

5322. (J. L. McKenzie, B.A.)—A triangle ABC is inscribed in a conic; through C a line is drawn parallel to the tangent at A, and meeting the conic in D; through A a line is drawn parallel to BC, and meeting the conic in E: prove that DE is parallel to AB.

5323. (Rev. E. Hill, M.A.)—The radius of a thermometer bulb is ·3 of an inch, that of the stem ·01, height of the freezing point above the bulb 1 inch, and of the boiling point $7\frac{1}{2}$ inches; show that the expansion of mercury for a degree centigrade is about $\frac{1}{5550}$ of its volume.

5324. (Dr. Hart.)—Find two integral numbers whose sum, difference, and difference of their squares shall *each* be a square, cube, and fourth power; and also the product of the nine roots of these powers shall be a square, cube, and fourth power.

5336. (The late T. Cotterill, M.A.)—Prove that the envelope of a limited line moving between two hyperbolas, with common asymptote is a class quartic (which is a correlative to the form mentioned by Dr. Salmon in his *Higher Plane Curves*, Art. 246, 2nd Ed.).

5343. (Lionel H. Rosenthal, M.A.)—Prove that (1) the six points of inflexion of a unicursal quartic lie on a conic; and (2) the six points in which the inflexional tangents meet the curve again also lie on a conic.

5345. (Dr. Hart.)—Find a triangle whose sides, perpendicular, line bisecting the vertical angle, and area, shall be integers.

5346. (W. E. Wright, B.A.)—If $\dfrac{\sin\alpha}{\sin(\alpha-2\beta)} = c$, find $\int \sin\alpha\, d\beta$.

5348. (J. Dawson.)—One side of a triangle given in species passes through a given point and is also a chord of a given conic; find the locus of the angle opposite that side.

5349. (A. Martin, LL.D.)—Find the mean distance of all the points in a groin from its centre.

5368. (H. W. Harris, B.A.)—Find the locus of the middle point of a chord of an ellipse, making any given angle with the tangent at its extremity; and point out some singularities of this curve.

5369. (R. Battle, M.A.)—Find the mean area of the triangle formed by joining a given point to the ends of a chord of a given ellipse, and also the greatest triangle of the series, the chord being of given length.

5370. (The Rev. W. Roberts, M.A.)—If

$$\tan^2\theta = \cos^2\phi + \cos^2\alpha \sin^2\phi \cos^2\psi + \cos^2\beta \sin^2\phi \sin^2\psi,$$

find the value of the double definite integral

$$\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{\theta \sin^2\phi \sin\psi\, d\phi\, d\psi}{\tan\theta}.$$

5375. (J. J. Walkor, F.R.S.)—If u is a ternary form of the order n, and if x, y, z satisfy the relation $\alpha x + \beta y + \gamma z - 0$, show how to transform

$$\alpha^2 x^4 \frac{d^2u}{dx^2} + \beta^2 y^4 \frac{d^2u}{dy^2} + \gamma^2 z^4 \frac{d^2u}{dz^2} - 2\beta\gamma y^2 z^2 \frac{d^2u}{dy\,dz} - 2\gamma\alpha z^2 x^2 \frac{d^2u}{dz\,dx} - 2\alpha\beta x^2 y^2 \frac{d^2u}{dy\,dx}$$

into

$$xyz\left\{2(x-1)\left(\beta\gamma\frac{du}{dx} + \gamma\alpha\frac{du}{dy} + \alpha\beta\frac{du}{dz}\right) - \left(\beta\gamma x\frac{d^2u}{dx^2} + \gamma\alpha y\frac{d^2u}{dy^2} + \alpha\beta z\frac{d^2u}{dz^2}\right)\right\}$$
$$-n(n-1)(\beta\gamma yz + \gamma\alpha zx + \alpha\beta xy)\,u.$$

5376. (The late T. Cotterill, M.A.)—1. If (x_1, y_1), (x_2, y_2) are the perpendiculars from two conjugate foci of a conic upon any two of its conjugate lines x and y, prove that $(x_1 y_1 + y_1 x_2) \sec (xy)$ is invariable.

2. Hence (or geometrically) show that conjugate foci of a conic touching CA, CB at A and B are foci of a conic touching AB and the reflexions of AB to CA and CB.

3. Prove that the same holds good for the sphere.

5377. (The Rev. A. F. Torry, M.A.)—Prove that the chord which joins the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ on the conic $l\alpha^{\frac{1}{2}} + m\beta^{\frac{1}{2}} + n\gamma^{\frac{1}{2}} = 0$ is parallel to

$$\frac{l\alpha}{\alpha_1^{\frac{1}{2}} + \alpha_2^{\frac{1}{2}}} + \frac{m\beta}{\beta_1^{\frac{1}{2}} + \beta_2^{\frac{1}{2}}} + \frac{n\gamma}{\gamma_1^{\frac{1}{2}} + \gamma_2^{\frac{1}{2}}} = 0.$$

5381. (Dr. Hart.)—Find thirty biquadrate numbers whose sum shall be a biquadrate number.

5404. (Dr. Hart.)—Find three numbers such that, if the sum of their cubes be either added to, or subtracted from, the square of each, the sums and remainders shall be squares.

5410. (Professor Hudson, M.A.)—A uniform sphere is dragged by a horizontal force along a homogeneous fluid of twice its density; find the velocity which must be kept up in order that one-fourth of the vertical diameter may be immersed.

5411. (A. Martin, LL.D.)—Find the probability that a random shot will hit a target a feet square at a distance of b feet.

5447. (The Rev. W. Roberts, M.A.)—The cone

$$x^2 \cot^2\alpha + y^2 \cot\beta - z^2 = 0$$

intersects the sphere $x^2+y^2+z^2-a^2=0$ in a sphero-conic. Show that the equation of the tubular surface, which is the envelope of a sphere of constant radius k, whose centre moves along this sphero-conic, is had by equating to zero the discriminant of the following cubic in λ,

$$\frac{4a^2\sin^2\alpha\, x^2}{P^2+4\lambda a^2\cos^2\alpha} + \frac{4a^2\sin^2\beta\, y^2}{P^2+4\lambda a^2\cos^2\beta} - \frac{z^3}{\lambda} = 1,$$

where $$P = x^2+y^2+z^2+a^2-k^2.$$

5448. (R. Battle, M.A.)—If the ends of a fixed chord in a given ellipse be joined to any point on the curve, find the average area and greatest value respectively of the quadrilateral cut from the triangle so formed by the diameter to which the given chord is an ordinate.

5469. (Hugh McColl, B.A.) — If $x_1 = -y$, $x_4 = \frac{1}{2}y^2z^{-1}$, $x_7 = 4a$, $y_4 = -(8az)^{\frac{1}{2}}$, $y_5 = -2z$, $y_6 = -4a$, $z_2 = 2a$, show that

$$p'\left(z-\frac{y^2}{2x}\right) = x_{4'.0}z_0 + x_0 z_{0'} + x_{4'}z_{0'} \quad\text{.......................(1)},$$

$$x_{7'.4'.1}y_{0'}z_0 = z_{2'.0}\left(y_{4'.6}x_{7'.1} + y_{5'.4}x_{4'.1}\right) \quad\text{....................(2)},$$

the symbols to be interpreted as in my articles on Symbolical Language.
[These steps are required in the solution of Quest. 5373.]

5473. (H. W. Harris, M.A.)—Prove that, if

$$F(m, n, p) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha^m & \beta^m & \gamma^m & \delta^m \\ \alpha^n & \beta^n & \gamma^n & \delta^n \\ \alpha^p & \beta^p & \gamma^p & \delta^p \end{vmatrix},$$

$$F(m, n, p)\left\{F\,1, 2, 3\right\}^2 = \begin{vmatrix} F(m, 2, 3), & F(1, m, 3), & F(1, 2, m) \\ F(n, 2, 3), & F(1, n, 3), & F(1, 2, n) \\ F(p, 2, 3), & F(1, p, 3), & F(1, 2, p) \end{vmatrix};$$

and generalize for a similar determinant of any order.

5485. (R. A. Roberts, M.A.)—If tangents are drawn to a cubic from any point of a harmonic polar, prove that they form a pencil in involution.

APPENDIX II.

SOLUTIONS OF QUESTIONS IN THE THEORY OF PROBABILITY AND AVERAGES.

By Professor G. B. M. Zerr, M.A.

11031. (Professor Zerr.)—To find the average distance between two points taken at random in the surface of a sector of a circle.

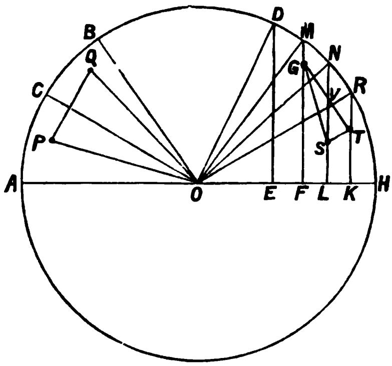
Solution.

Let P, Q be the random points; also put

$$OA = r, \quad OP = y, \quad OQ = x,$$

$$\angle AOB = \beta, \quad \angle AOQ = \theta, \quad \angle AOP = \phi.$$

Then an element of the sector at P is $y\,dy\,d\phi$, at Q, $x\,dx\,d\theta$.



The limits of θ are 0 and β; of ϕ, 0 and θ; of x, 0 and r; and of y, 0 and x.

$$PQ = u = \left[x^2+y^2-2xy\cos(\theta-\phi)\right]^{\frac{1}{2}}.$$

If Δ = the required average, then

$$\Delta = \frac{\int_0^\beta\int_0^\theta\int_0^r\int_0^x u\,x\,dx\,y\,dy\,d\theta\,d\phi}{\int_0^\beta\int_0^\theta\int_0^r\int_0^x x\,dx\,y\,dy\,d\theta\,d\phi}$$

$$= \frac{16}{\beta^2 r^4}\int_0^\beta\int_0^\theta\int_0^r\int_0^x \left[x^2+y^2-2xy\cos(\theta-\phi)\right]^{\frac{1}{2}} xy\,d\theta\,d\phi\,dx\,dy$$

$$= \frac{8}{3\beta^2\gamma^4}\int_0^\beta\int_0^\theta\int_0^r \Big[16\sin^3\tfrac{1}{2}(\theta-\phi)+12\sin^3\tfrac{1}{2}(\theta-\phi)\cos(\theta-\phi)-2$$
$$+3\cos^2(\theta-\phi)+3\sin^2(\theta-\phi)\cos(\theta-\phi)$$
$$\log\left\{1+\operatorname{cosec}\tfrac{1}{2}(\theta-\phi)\right\}\Big]\,x^4\,d\theta\,d\phi\,dx$$

$$= \frac{8r}{15\beta^2}\int_0^\beta\int_0^\theta \Big[16\sin^3\tfrac{1}{2}(\theta-\phi)+12\sin^3\tfrac{1}{2}(\theta-\phi)\cos(\theta-\phi)-2$$
$$+3\cos^2(\theta-\phi)+3\sin^3(\theta-\phi)\cos(\theta-\phi)$$
$$\log\left\{1+\operatorname{cosec}\tfrac{1}{2}(\theta-\phi)\right\}\Big]\,d\theta\,d\phi$$

$$= \frac{4r}{45\beta^2}\int_0^\beta \Big[48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+12\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-32\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$
$$-6\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-64\cos\tfrac{1}{2}\theta+64+9\sin\theta\cos\theta$$
$$+6\sin^3\theta\log(1+\operatorname{cosec}\theta)\Big]\,d\theta$$

$$= \frac{2r}{135\beta^2}\Big[16\sin^6\tfrac{1}{2}\beta+16\cos^6\tfrac{1}{2}\beta+96\sin^5\tfrac{1}{2}\beta+24\sin^4\tfrac{1}{2}\beta-12\cos^4\tfrac{1}{2}\beta$$
$$-112\sin^3\tfrac{1}{2}\beta-36\sin^2\tfrac{1}{2}\beta-720\sin\tfrac{1}{2}\beta+27\sin^2\beta+384\beta$$
$$-4-12(\sin^2\beta\cos\beta+2\cos\beta+2)\log(1+\sin\tfrac{1}{2}\beta)$$
$$+12(\sin^2\beta\cos\beta+2\cos\beta-2)\log\sin\tfrac{1}{2}\beta\Big].$$

When $\beta = 2\pi$, we have $\qquad \Delta = \dfrac{128r}{45\pi}$;

when $\beta = \pi$, we have $\qquad \Delta = \dfrac{256r}{45\pi} - \dfrac{1472r}{135\pi^2}$

(see *Mathematical Visitor*, Vol. I., No. 3, p. 70);

when $\beta = \frac{1}{2}\pi$, we have

$$\Delta = \frac{32r}{135\pi}\left(48+\frac{3}{\pi}-\frac{94\sqrt{2}}{\pi}-\frac{6}{\pi}\log\frac{1+\sqrt{2}}{2}\right).$$

As a check for the last value of Δ we have the following, from WILLIAMSON's *Integral Calculus*:—

$$\Delta_1+\Delta = 2\Delta_2 \quad\text{.................................(1)},$$

$$\Delta_1 = \frac{32r}{135\pi^2}\left[94\sqrt{2}-95+6\log\left(\frac{1+\sqrt{2}}{2}\right)\right]$$

(*Visitor*, Vol. I., No. 2, p. 36),

$$\Delta_2 = \frac{256r}{45\pi}-\frac{1472r}{135\pi^2} \quad (\textit{Visitor}, \text{Vol. I., No. 3, p. 70}).$$

Substituting these values of Δ_1 and Δ_2 in (1), we find Δ the same as given above.

This general value β will give the average distance between two random points in any sector between 0 and 360°.

[Prof. ZERR remarks that this general proof was suggested to him by Dr. ARTEMUS MARTIN's solution of Problem 73, p. 70, Vol. I., No. 3, of the *Mathematical Visitor*.]

11032. (Professor ZERR.)—To find the average area of the triangle formed by joining the two points P and Q, in a circular sector, with the centre O.

Solution.

Let Δ_0 be the required average. Then, with the same notation as in Quest. 11031, since the area is $\frac{1}{2}xy\sin(\theta-\phi)$, we get

$$\Delta_0 = \frac{\int_0^\beta\int_0^\theta\int_0^r\int_0^x \frac{1}{2}xy\sin(\theta-\phi)\,x\,dx\,y\,dy\,d\theta\,d\phi}{\int_0^\beta\int_0^\theta\int_0^r\int_0^x xy\,d\theta\,d\phi\,dx\,dy}$$

$$= \frac{8}{\beta^2r^4}\int_0^\beta\int_0^\theta\int_0^r\int_0^x x^2y^2\sin(\theta-\phi)\,d\theta\,d\phi\,dx\,dy$$

$$= \frac{8}{3\beta^2r^4}\int_0^\beta\int_0^\theta\int_0^r x^5\sin(\theta-\phi)\,d\theta\,d\phi\,dx$$

$$= \frac{4r^2}{9\beta^2}\int_0^\beta\int_0^\theta \sin(\theta-\phi)\,d\theta\,d\phi$$

$$= \frac{4r^2}{9\beta^2}\int_0^\beta (1-\cos\theta)\,d\theta$$

$$= \frac{4r^2}{9\beta^2}(\beta-\sin\beta);$$

when $\beta = 2\pi$, we have $\Delta_0 = \dfrac{2r^2}{9\pi}$;

when $\beta = \pi$, we have $\Delta_0 = \dfrac{4r^2}{9\pi}$;

when $\beta = \frac{1}{2}\pi$, we have $\Delta_0 = \dfrac{8r^2}{9\pi^2}(\pi-2)$.

11033. (Professor ZERR.)—To find the average distance between two points taken at random in any sector, but one on each side of the diameter bisecting the angle of the sector.

Solution.

Let OC be the radius bisecting the angle of the sector AOB.

Let $OQ = x$, $OP = y$, $\angle AOB = 2\beta$, $\angle COP = \phi$, $\angle COQ = \theta$. The limits of x are 0 and r; of y, 0 and x; θ, 0 and β; ϕ, 0 and β.

$$PQ = u = \left[x^2+y^2-2xy\cos(\theta+\phi)\right]^{\frac{1}{2}},$$

M = required average; then

$$M=\frac{\int_0^\beta\int_0^\beta\int_0^r\int_0^x u\,d\theta\,d\phi\,x\,dx\,y\,dy}{\int_0^\beta\int_0^\beta\int_0^r\int_0^x d\theta\,d\phi\,x\,dx\,y\,dy}$$

$$=\frac{8}{\beta^2r^4}\int_0^\beta\int_0^\beta\int_0^r\int_0^x\left[x^2+y^2-2xy\cos(\theta+\phi)\right]\,d\theta\,d\phi\,x\,dx\,y\,dy$$

$$=\frac{2}{3\beta^2r^4}\int_0^\beta\int_0^\beta\int_0^r\Big[8\sin\tfrac12(\theta+\phi)+40\sin\tfrac12(\theta+\phi)\cos^2\tfrac12(\theta+\phi)$$
$$-48\sin\tfrac12(\theta+\phi)\cos^4\tfrac12(\theta+\phi)+3\cos2(\theta+\phi)-1$$
$$+6\sin^2(\theta+\phi)\cos(\theta+\phi)\log\left\{1+\operatorname{cosec}\tfrac12(\theta+\phi)\right\}\Big]$$
$$\times\,d\theta\,d\phi\,x^4dx$$

$$=\frac{2r}{15\beta^2}\int_0^\beta\int_0^\beta\Big[8\sin\tfrac12(\theta+\phi)+40\sin\tfrac12(\theta+\phi)\cos^2\tfrac12(\theta+\phi)$$
$$-48\sin\tfrac12(\theta+\phi)\cos^4\tfrac12(\theta+\phi)+3\cos2(\theta+\phi)-1$$
$$+6\sin^2(\theta+\phi)\cos(\theta+\phi)\log\left\{1+\operatorname{cosec}\tfrac12(\theta+\phi)\right\}\Big]\,d\theta\,d\phi$$

$$=\frac{2r}{45\beta^2}\int_0^\beta\Big[48\cos\tfrac12\theta-48\cos\tfrac12(\theta+\beta)+64\cos^3\tfrac12\theta-64\cos^3\tfrac12(\theta+\beta)$$
$$-48\cos^5\tfrac12\theta+48\cos^5\tfrac12(\theta+\beta)-3\sin2\theta+3\sin2(\theta+\beta)$$
$$-6\sin^3\theta\log(1+\operatorname{cosec}\tfrac12\theta)$$
$$+6\sin^3(\theta+\beta)\log\left\{1+\operatorname{cosec}\tfrac12(\theta+\beta)\right\}\Big]\,d\theta$$

$$=\frac{2r}{1350\beta^2}\Big\{7328\sin\tfrac12\beta-3664\sin\beta+1024\sin\tfrac12\beta\cos^2\tfrac12\beta-512\sin\beta\cos^2\beta$$
$$-1152\sin\tfrac12\beta\cos^4\tfrac12\beta+576\sin\beta\cos^4\beta-45\cos4\beta+90\cos2\beta$$
$$-160\sin^3\tfrac12\beta+80\sin^3\beta+192\sin^5\tfrac12\beta-96\sin^5\beta+120\sin^4\tfrac12\beta$$
$$-60\sin^4\beta-160\sin^6\tfrac12\beta+80\sin^6\beta-160\cos^6\tfrac12\beta+80\cos^6\beta$$
$$-25+120\cos^4\tfrac12\beta-60\cos^4\beta$$
$$-120(\sin^2\beta\cos\beta+2\cos\beta-2)\log\sin\tfrac12\beta$$
$$+120(\sin^2\beta\cos\beta+2\cos\beta+2)\log(1+\sin\tfrac12\beta)$$
$$+60(\sin^22\beta\cos2\beta+2\cos2\beta-2)\log\sin\beta$$
$$-60(\sin^22\beta\cos2\beta+2\cos2\beta+2)\log(1+\sin\beta)\Big\}.$$

When $\beta=\pi$, the average becomes

$$M=\frac{1472r}{135\pi^2}$$

(see Problem 29, No. 2, Vol. I., *Mathematical Visitor*);

when $\beta=\frac12\pi$, we have

$$M=\frac{32r}{135\pi^2}\left\{94\sqrt2-95+6\log\frac{1+\sqrt2}{2}\right\}$$

(see Problem 32 of *Visitor*).

11034. (Professor ZERR).—To find the average area of the triangle formed by joining the two points in Question 11033 to the centre O.

Solution.

The area of the triangle is $\frac{1}{2}xy\sin(\theta+\phi)$. If M_1 = the required average, we have, with the same notation as in the last Question (11034),

$$M_1 = \frac{\int_0^\beta\int_0^\beta\int_0^r\int_0^x \frac{1}{2}xy\sin(\theta+\phi)\,d\theta\,d\phi\,x\,dx\,y\,dy}{\int_0^\beta\int_0^\beta\int_0^r\int_0^x d\theta\,d\phi\,x\,dx\,y\,dy}$$

$$= \frac{4}{\beta^2 r^4}\int_0^\beta\int_0^\beta\int_0^r\int_0^x x^2y^2\sin(\theta+\phi)\,d\theta\,d\phi\,dx\,dy$$

$$= \frac{4}{3\beta^2 r^4}\int_0^\beta\int_0^\beta\int_0^r x^5\sin(\theta+\phi)\,d\theta\,d\phi\,dx$$

$$= \frac{2r^2}{9\beta^2}\int_0^\beta\int_0^\beta \sin(\theta+\phi)\,d\theta\,d\phi$$

$$= \frac{2r^2}{9\beta^2}\int_0^\beta \left[\cos\theta-\cos(\theta+\beta)\right]d\theta$$

$$= \frac{4r^3}{9\beta^2}\sin\beta\,(1-\cos\beta).$$

When $\beta = \frac{1}{2}\pi$, we have $\quad M_1 = \frac{16r^2}{9\pi^2}$;

when $\beta = \pi$, we have $\quad M_1 = 0$.

11035. (Professor ZERR.)—To find the average area of a triangle formed by joining three points taken at random in that part of a sector which is included between its arc DH, the perpendicular DE, and the side EH.

Solution.

Let G, S, T be the three random points.

Through G, S, T, draw MF, NL, RK perpendicular to OH, NL intersecting GT in V.

Let $OH = r$, $FG = x$, $KT = y$, $LS = z$, $FM = x'$, $KR = y'$, $LN = z'$, $LV = z''$, $\angle FOM = \theta$, $\angle KOR = \phi$, $\angle LON = \psi$, $\angle HOD = \beta$, $v = \dfrac{1}{\cos\phi - \cos\phi}$.

Then we have $\quad x' = r\sin\theta,\quad y' = r\sin\phi,\quad z' = r\sin\psi,$

$$z'' = v\left\{x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)\right\}.$$

$$\text{Area GST} = \tfrac{1}{2}r\left[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)+z(\cos\theta-\cos\phi)\right] = U,$$
when $z < z''$;

$$\text{Area GST} = \tfrac{1}{2}r\left[x(\cos\psi-\cos\phi)+y(\cos\theta-\cos\psi)+z(\cos\phi-\cos\theta)\right] = U_1,$$
when $z > z''$.

An element of surface at T is $r \sin\phi\, d\phi\, dy$, at S it is $r\sin\psi\, d\psi\, dz$, and at G it is $r\sin\theta\, d\theta\, dx$.

The limits of θ are 0 and β; of ϕ, 0 and ϕ; of ψ, ϕ and θ; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'', and z'' and z'.

The average area of the triangle is

$$M_1 = \frac{\int_0^\beta\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\left[\int_0^{z''} u\,dz + \int_{z''}^{z'} u_1\,dz\right] r\sin\theta d\theta\, r\sin\phi d\phi\, r\sin\psi d\psi\, dx\, dy}{\int_0^\beta\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\int_0^{z'} r\sin\theta d\theta\, r\sin\phi d\phi\, r\sin\psi d\psi\, dx\, dy\, dz}.$$

Integrating numerator,

$$\int_0^\beta\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\left[\int_0^{z''} u\,dz + \int_{z''}^{z'} u_1\,dz\right] r\sin\theta d\theta\, r\sin\phi d\phi\, r\sin\psi d\psi\, dx\, dy$$

$$= \frac{r^4}{4}\int_0^\beta\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\Big\{\left[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)\right]^2$$
$$+\left[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)\right.$$
$$\left.+r\sin\psi(\cos\theta-\cos\phi)\right]^2\Big\}\sin\theta\sin\phi\sin\psi v\, d\theta\, d\phi\, d\psi\, dx\, dy$$

$$= \frac{r^5}{12}\int_0^\beta\int_0^\theta\int_\phi^\theta\int_0^{x'}\left[6x^2\sin\phi(\cos\phi-\cos\psi)^2+6rx\sin^2\phi(\cos\phi-\cos\psi)\right.$$
$$\times(\cos\psi-\cos\theta)+2r^2\sin^3\phi(\cos\psi-\cos\theta)^2$$
$$+6rx\sin\phi\sin\psi(\cos\phi-\cos\psi)(\cos\theta-\cos\phi)+3r^2\sin\phi\sin^2\psi(\cos\theta-\cos\phi)^2$$
$$\left.+3r^2\sin^2\phi\sin\psi(\cos\theta-\cos\phi)(\cos\psi-\cos\theta)\right]\sin\theta\sin\phi\sin\psi v\, d\theta\, d\phi\, d\psi\, dx$$

$$= \frac{r^8}{12}\int_0^\beta\int_0^\theta\int_\phi^\theta\left[2\sin^3\theta\sin\phi(\cos\phi-\cos\psi)^2+2\sin\theta\sin^3\phi(\cos\psi-\cos\theta)^2\right.$$
$$+3\sin^2\theta\sin^2\phi(\cos\phi-\cos\psi)(\cos\psi-\cos\theta)$$
$$+3\sin\theta\sin\phi\sin^2\psi(\cos\theta-\cos\phi)^2$$
$$+\sin^2\theta\sin\phi\sin\psi(\cos\phi-\cos\psi)(\cos\theta-\cos\phi)$$
$$\left.+3\sin\theta\sin^2\phi\sin\psi(\cos\psi-\cos\theta)(\cos\theta-\cos\phi)\right]$$
$$\times\sin\theta\sin\phi\sin\psi v\, d\theta\, d\phi\, dy$$

$$= \frac{r^8}{72}\int_0^\beta\int_0^\theta\left[4\sin^2\theta\cos^2\phi+4\sin^2\phi\cos^2\phi\right.$$
$$+4\sin^2\theta\cos^2\theta+4\sin^2\phi\cos^2\theta+\sin^2\theta\cos\theta\cos\phi$$
$$+\sin^2\phi\cos\phi\cos\theta-6\sin\theta\cos\theta\sin\phi\cos\phi+6\cos^3\theta\cos\phi$$
$$+6\cos\theta\cos^3\phi+12+6\cos^2\theta+6\cos^2\phi-36\cos\theta\cos\phi$$
$$-12\sin\theta\sin\phi-9(\theta-\phi)\sin\theta\cos\phi$$
$$\left.+9(\theta-\phi)\sin\phi\cos\theta\right]\sin^2\theta\sin^2\phi\, d\theta\, d\phi$$

$$= \frac{r^8}{432}\int_0^\beta\Big(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$
$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta\Big)\sin^2\theta\, d\theta$$

$$= \frac{r^8}{3456}\Big(105\beta^2-105\beta\sin2\beta+44\beta\sin^3\beta\cos\beta+16\beta\sin^5\beta\cos\beta+96\beta\sin^3\beta$$
$$-384+105\sin^2\beta-101\sin^4\beta-16\sin^6\beta+3\sin^8\beta$$
$$+192\sin^2\beta\cos\beta+384\cos\beta\Big).$$

Integrating the denominator,

$$\int_0^\beta \int_0^\theta \int_\phi^\theta \int_0^{x'} \int_0^{y'} \int_0^{z'} r\sin\theta\, d\theta\, r\sin\phi\, d\phi\, r\sin\psi\, d\psi\, dx\, dy\, dz$$

$$= r^6 \int_0^\beta \int_0^\theta \int_\phi^\theta \sin^2\theta \sin^2\phi \sin^2\psi\, d\theta\, d\phi\, d\psi$$

$$= \frac{r^6}{2} \int_0^\beta \int_0^\theta (\theta - \sin\theta\cos\theta - \phi + \sin\phi\cos\phi) \sin^2\theta \sin^2\phi\, d\theta\, d\phi$$

$$= \frac{r^6}{8} \int_0^\beta (\theta^2 - 2\theta\sin\theta\cos\theta + \sin^2\theta - \sin^4\theta) \sin^2\theta\, d\theta$$

$$= \frac{r^6}{48} (\beta - \sin\beta\cos\beta)^3.$$

Hence

$$M_1 = \frac{r^2}{72\,(\beta - \sin\beta\cos\beta)^3} \Big(105\beta^2 - 105\beta\sin 2\beta + 44\beta\sin^3\beta\cos\beta + 16\beta\sin^5\beta\cos\beta + 96\sin^3\beta - 384 + 105\sin^2\beta - 101\sin^4\beta - 16\sin^6\beta + 3\sin^8\beta + 192\sin^2\beta\cos\beta + 384\cos\beta\Big).$$

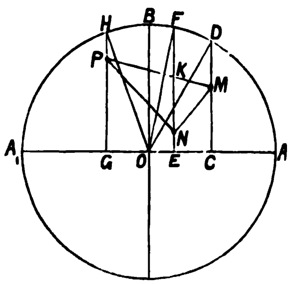
When $\beta = 2\pi$, we have $\quad M_1 = \dfrac{35r^2}{48\pi}$;

when $\beta = \pi$, we have $\quad M_1 = \dfrac{r^2}{\pi}\left(\dfrac{35}{24} - \dfrac{32}{3\pi^2}\right)$;

when $\beta = \frac{1}{2}\pi$, we have $\quad M_1 = \dfrac{r^2}{\pi}\left(\dfrac{35}{12} + \dfrac{16}{3\pi} - \dfrac{131}{3\pi^2}\right)$.

11036. (Professor ZERR.)—To find the average area of the triangle formed by joining three points taken at random, (1) one in each quadrant and the third anywhere in the semicircle, (2) one in each semicircle and the third anywhere in the circle.

Solution.



Let M be the random point in the quadrant BOA, P in the quadrant BOA_1, N anywhere in the semicircle ABA_1. Through M, N, P, draw DC, EF, GH_1 perpendicular to AA_1, EF intersecting MP at K.

Let $OA = r$, $GP = x$, $CM = y$, $EN = z$, $\angle AOH = \theta$, $\angle AOD = \phi$, $\angle AOF = \psi$, $GH = r\sin\theta = x'$, $CD = r\sin\phi = y'$, $EF = r\sin\psi = z'$,

and $\quad V = \dfrac{1}{\cos\phi - \cos\theta}$.

$$EK = V\left\{x(\cos\phi - \cos\psi) + y(\cos\psi - \cos\theta)\right\} = z''.$$

Area MNP $= \frac{1}{2}r\left[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)+z(\cos\theta-\cos\phi)\right]=u$,
when $z < z''$;

Area MNP $= \frac{1}{2}r\left[x(\cos\psi-\cos\phi)+y(\cos\theta-\cos\psi)+z(\cos\phi-\cos\theta)\right]=u_1$,
when $z > z''$.

An element of the surface at M is $r\sin\phi\, d\phi\, dy$; at N it is $r\sin\psi\, d\psi\, dz$; at P, $r\sin\theta\, d\theta\, dx$.

When the points are taken as in (1), the limits of θ are $\frac{1}{2}\pi$ and π; of ϕ, 0 and $\frac{1}{2}\pi$; of ψ, ϕ and θ; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'', and z'' and z'.

Hence the average area is

$$\Delta = \frac{\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi}\int_\phi^\theta\int_0^{x'}\int_0^{y'}\left[\int_0^{z''}u\,dz+\int_{z''}^{z'}u_1 dz\right] r\sin\theta\, d\theta\, r\sin\phi\, d\phi\, r\sin\psi\, d\psi\, dx\, dy}{\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi}\int_\phi^\theta\int_0^{x'}\int_0^{y'}\int_0^{z'} r\sin\theta\, d\theta\, r\sin\phi\, d\phi\, r\sin\psi\, d\psi\, dx\, dy\, dz}$$

$$= \frac{64}{\pi^3 r^3}\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi}\int_\phi^\theta\int_0^{x'}\int_0^{y'}\left[\int_0^{z''}u\,dz+\int_{z''}^{z'}u_1 dz\right]\sin\theta\sin\phi\sin\psi\, d\theta\, d\phi\, d\psi\, dx\, dy.$$

All the integrations in this have been performed, except the last two, in the preceding example. Hence

$$\begin{aligned}\Delta = \frac{8r^2}{9\pi^3}\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi}\big[&4\sin^2\theta\cos^2\phi+4\sin^2\phi\cos^2\theta+4\sin^2\phi\cos^2\phi+4\sin^2\theta\cos^2\theta\\ &+\sin^2\theta\cos\theta\cos\phi+\sin^2\phi\cos\phi\cos\theta-6\sin\theta\cos\theta\sin\phi\cos\phi\\ &+6\cos^3\phi\cos\theta+6\cos\phi\cos^3\theta+12+6\cos^2\theta+6\cos^2\phi\\ &-36\cos\theta\cos\phi-12\sin\theta\sin\phi-9(\theta-\phi)\sin\theta\cos\phi\\ &+9(\theta-\phi)\sin\phi\cos\theta\big]\sin^2\theta\sin^2\phi\, d\theta\, d\phi;\end{aligned}$$

$$\begin{aligned}\Delta = \frac{2r^2}{9\pi^3}\int_{\frac{1}{2}\pi}^{\pi}\Big(&23\pi\sin^2\theta-4\pi\sin^4\theta-4\pi\sin^6\theta+6\pi\sin^3\theta-12\theta\sin^3\theta\\ &+24\theta\sin^2\theta\cos\theta-72\sin^2\theta\cos\theta-40\sin^3\theta-6\sin^3\theta\cos\theta\\ &+8\sin^2\theta\cos^3\theta\Big)\,d\theta\end{aligned}$$

$$= \frac{r^2}{\pi}\left\{\frac{35}{36}-\frac{16}{9\pi}+\frac{53}{135\pi^2}\right\}.$$

In (2) the limits of θ are π and 2π; of ϕ, 0 and π; and for the other variables the same as in (1). The number of ways three points can be taken as in (2) is eight times the number of ways they can be taken as in (1). Hence the required average is

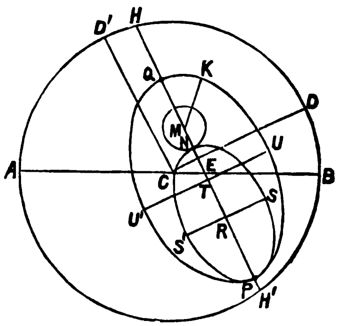
$$\begin{aligned}\Delta_1 = \frac{r^2}{9\pi^3}\int_\pi^{2\pi}\int_0^\pi\big[&4\sin^2\theta\cos^2\phi+4\cos^2\theta\sin^2\phi+4\sin^2\phi\cos^2\phi+4\sin^2\theta\cos^2\theta\\ &+\sin^2\theta\cos\theta\cos\phi+\sin^2\phi\cos\phi\cos\theta-6\sin\phi\cos\phi\sin\theta\cos\theta\\ &+6\cos^3\phi\cos\theta+6\cos\phi\cos^3\theta+12+6\cos^2\theta+6\cos^2\phi\\ &-36\cos\theta\cos\phi-12\sin\theta\sin\phi-9(\theta-\phi)\sin\theta\cos\phi\\ &+9(\theta-\phi)\sin\phi\cos\theta\big]\sin^2\theta\sin^2\phi\, d\theta\, d\phi\end{aligned}$$

$$\begin{aligned}= \frac{r^2}{18\pi^3}\int_\pi^{2\pi}\Big(&23\pi\sin^2\theta-4\pi\sin^4\theta-4\pi\sin^6\theta-40\sin^3\theta+24\theta\sin^2\theta\cos\theta\\ &-12\pi\sin^2\theta\cos\theta\Big)\,d\theta\end{aligned}$$

$$= \frac{r^2}{\pi}\left\{\frac{35}{72}+\frac{32}{9\pi^2}\right\}.$$

11037. (Professor ZERR.)—Two points are taken at random in the surface of a given circle. An ellipse is described on the distance between the two points as major axis. If a point be taken at random in the left-hand half of this major axis, and with this point as centre a circle is described at random, but so as to lie wholly within the ellipse, find the average area of the ellipse described on that portion of the major axis between the right-hand extremity and the circumference of the random circle.

Solution.



Let P, Q be the random points, PQ the major axis of the ellipse QUPU′. Through P, Q draw HH′. Draw CD, CD′ perpendicular and parallel to HH′. Let M be the centre of the random circle.

Let $AC = r$, $CE = w$,

$HE = EH' = u$, $HP = x$,

$PQ = y$, $\angle D'CA = \theta$, $QM = z$,

$MN = s$, $TU = \rho$,

$NR = v$, $RS = n$.

Then we have $u^2 = (r^2 - w^2)$,

$NR = \frac{1}{2}(y - z - s)$,

$$MK = s_1 = \rho\left(1 - \frac{(\frac{1}{2}y - z)^2}{\frac{1}{4}y^2 - \rho^2}\right)^{\frac{1}{2}},$$

$z_1 = \dfrac{\rho^2}{\frac{1}{2}y}$ = radius of curvature at Q, area $NSPS' = \pi vn$.

An element of the circle at P is $dw\,dx$, at Q it is $y\,d\theta\,dy$.

The limits of θ are 0 and $\frac{1}{2}\pi$ and doubled; of w, $-r$ and $+r$; of x, 0 and $2u$; of y, 0 and x and doubled; of n, 0 and v; of s, 0 and z when z is less than z_1, and 0 and s_1 when z is greater than z_1; of z, 0 and $\frac{1}{2}y = y_1$; of ρ, 0 and $\frac{1}{2}y = y_1$.

Hence

$$\Delta = \frac{\displaystyle\int_0^{\frac{1}{2}\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x}\int_0^{y_1}\left[\int_0^{z_1}\int_0^{z}\int_0^{v}\pi vn\,dz\,ds\,dn + \int_{z_1}^{y_1}\int_0^{s_1}\int_0^{v}\pi vn\,dz\,ds\,dn\right] \times y\,d\theta\,du\,dx\,dy\,d\rho}{\displaystyle\int_0^{\frac{1}{2}\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x}\int_0^{y_1}\left[\int_0^{z_1}\int_0^{z}\int_0^{v}dz\,ds\,dn + \int_{z_1}^{y_1}\int_0^{s_1}\int_0^{v}dz\,ds\,dn\right] y\,d\theta\,dw\,dx\,dy\,d\rho}$$

$$= \frac{1}{8}\pi\,\frac{\displaystyle\int_0^{\frac{1}{2}\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x}\int_0^{y_1}\left[\int_0^{z_1}\int_0^{z}(2y_1 - z - s)^3\,dz\,ds + \int_{z_1}^{y_1}\int_0^{s_1}(2y_1 - z - s)^3\,dz\,ds\right] \times y\,d\theta\,dw\,dx\,dy\,d\rho}{\displaystyle\int_0^{\frac{1}{2}\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x}\int_0^{y_1}\left[\int_0^{z_1}\int_0^{z}(2y_1 - z - s)\,dz\,ds + \int_{z_1}^{y_1}\int_0^{s_1}(2y_1 - z - s)^3\,dz\,ds\right] \times y\,d\theta\,dw\,dx\,dy\,d\rho}$$

$$\Delta = \frac{1}{16}\pi \frac{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x}\int_0^{y_1}\left[\int_0^{z_1}(32y_1^3z-72y_1^2z^2+56y_1z^3-15z^4)\,dz + \int_{z_1}^{y_1}\left\{(2y_1-z)^4-(2y_1-z-s_1)^4\right\}dz\right] y\,d\theta\,dw\,dx\,dy\,d\rho}{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x}\int_0^{y_1}\left[\int_0^{z_1}(4y_1z-3z^2)\,dz + \int_{z_1}^{y_1}\left\{(2y_1-z)^2-(2y_1-z-s_1)^2\right\}dz\right] y\,d\theta\,du\,dx\,dy\,d\rho}$$

$$= \frac{1}{160}\pi \frac{\displaystyle\int_0^{\frac12\pi}\int_{+r}^{+r}\int_0^{2u}\int_0^{x}\int_0^{y_1}\left[105y_1^3\rho\,(y_1^2-\rho^2)^{\frac12}\cos^{-1}\frac{\rho}{y_1}+136y_1^4\rho-129y_1^3\rho^2 -128y_1^2\rho^3+295y_1\rho^4-8\rho^5-\frac{96\rho^6}{y_1}+\frac{16\rho^8}{y_1^3}\right] y\,d\theta\,dw\,dx\,dy\,d\rho}{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x}\int_0^{y_1}\left[y_1\rho\,(y_1^2-\rho^2)^{\frac12}\cos^{-1}\frac{\rho}{y_1}+2y_1^2\rho +y_1\rho^2-2\rho^3+\frac{2\rho^4}{y_1}\right] y\,d\theta\,dw\,dx\,dy\,d\rho}$$

$$= \frac{\pi}{2688}\left(\frac{2205\pi+2012}{15\pi+17}\right)\cdot\frac{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x} y^7\,d\theta\,dw\,dx\,dy}{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r}\int_0^{2u}\int_0^{x} y^5\,d\theta\,dw\,dx\,dy}$$

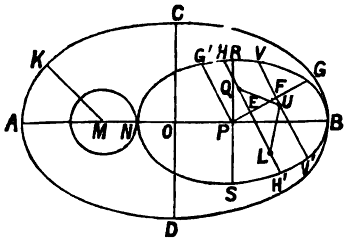
$$= \frac{\pi}{3584}\left(\frac{2205\pi+2012}{15\pi+17}\right)\cdot\frac{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r}\int_0^{2u} x^8\,d\theta\,dw\,dx}{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r}\int_0^{2u} x^6\,d\theta\,dw\,dx}$$

$$= \frac{\pi}{1152}\left(\frac{2205\pi+2012}{15\pi+17}\right)\cdot\frac{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r} u^9\,d\theta\,dw}{\displaystyle\int_0^{\frac12\pi}\int_{-r}^{+r} u^7\,d\theta\,dw}$$

$$= \frac{\pi r^2}{1280}\left(\frac{2205\pi+2012}{15\pi+17}\right).$$

11038. (Professor Zerr.)—From a point taken at random in the left-hand half of the major axis ($=2a$) of an ellipse whose minor axis is unknown, a circle is drawn at random, but so as to lie wholly in the surface of the ellipse; find the average area of the triangle formed by taking three points at random in the ellipse whose major axis is that portion of the given major axis between its right-hand extremity and the circumference of the circle.

Solution.



Let ACBD be the ellipse whose major axis AB $= 2a$, but whose minor axis CD is unknown; M the centre of the random circle; Q, U, L the three random points in the ellipse NRBS. Through Q, L draw the chord HH′, and through U draw the chord VV′ parallel to HH′. Draw PG′ parallel to HH′ and PG conjugate to PG′.

Let OC $= w$, AM $= x$,

MN $= y$, PR $= z$, NP $= v$, PG $= v'$, PG′ $= z'$, PE $= n$,

HL $= s$, QL $= r$, PF $= m$, HE $=$ EH′ $= u$, VF $=$ FV′ $= l$,

$\angle$ G′PN $= \theta$, $\angle$ PEH $= \phi$.

Then we have
$$\text{MK} = y_1 = w\left(1 - \frac{(a-x)^2}{a^2-w^2}\right)^{\frac{1}{2}},$$

$$x_1 = \frac{w^2}{a} = \text{radius of curvature at A},$$

$$v = \tfrac{1}{2}(2a - x - y), \qquad z'^2 = \frac{z^2}{1 - e^2\cos^2\theta},$$

$$u^2 = \frac{z'^2}{v'^2}(v'^2 - n^2), \qquad l^2 = \frac{z'^2}{v'^2}(v'^2 - m^2),$$

$$\sin\phi = \frac{vz}{v'z'}, \text{ and area QLU} = \tfrac{1}{2}(m-n)\, r\sin\phi.$$

An element of the ellipse at L is $\sin\phi\, dn\, ds$, at Q it is $r\, d\theta\, dr$, and at U it is $2l\sin\phi\, dm$. The limits of z are 0 and v; those of y are 0 and x when x is less than x_1, and 0 and y_1 when x is greater than x_1; those of x are 0 and a; of u, 0 and a; of θ, 0 and $\frac{1}{2}\pi$ and doubled; of n, $-v'$ and $+v'$; of s, 0 and $2u$; of r, 0 and s and doubled; and of m, n and v' and doubled. Hence, since the whole number of ways the three points can be taken, $\pi^3 v^3 z^3$, the required average area is

$$\Delta = \left\{\int_0^a \left[\int_0^{x_1}\int_0^x\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\int_0^s\int_n^{v'} (m-n)\sin^3\phi\, r^2\, l\, dx\, dy\, dz\, d\theta\, dn\, ds\, dr\, dm\right.\right.$$
$$\left.\left. + \int_{x_1}^a\int_0^{y_1}\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\int_0^s\int_0^{v'} (m-n)\sin^3\phi\, r^2\, l\, dx\, dy\, dz\, d\theta\, dn\, ds\, dr\, dm\right] dw\right\}$$
$$\div \int_0^a \left[\int_0^{x_1}\int_0^x\int_0^v \pi^3 v^3 z^3\, dx\, dy\, dz + \int_{x_1}^a\int_0^{y_1}\int_0^v \pi^3 v^3 z^3\, dx\, dy\, dz\right] dw.$$

Integrating the numerator, we obtain

$$8\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\int_0^s\int_n^{v'}(m-n)\sin^3\phi r^2 l\,dx\,dy\,dz\,d\theta\,dn\,ds\,dr\,dm\right.$$

$$\left.+\int_{x_1}^a\int_0^{y_1}\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\int_0^s\int_n^{v'}(m-n)\sin^3\phi r^2 l\,dx\,dy\,dz\,d\theta\,dn\,ds\,dr\,dm\right]dw$$

$$=\frac{2}{3}\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v dx\,dy\,dz+\int_{x_1}^a\int_0^{y_1}\int_0^v dx\,dy\,dz\right]dw\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\int_0^s\frac{z'}{v'}$$

$$\left\{4(v'^2-n^2)^{\frac{3}{2}}+6n^2(v'^2-n^2)^{\frac{1}{2}}-3\pi v'^2n+6v'^2n\sin^{-1}\frac{n}{v'}\right\}\sin^3\phi r^2\,d\theta\,dn\,ds\,dr$$

$$=\frac{2}{9}\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v dx\,dy\,dz+\int_{x_1}^a\int_0^{y_1}\int_0^v dx\,dy\,dz\right]dw\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\frac{z'}{v'}$$

$$\left\{4(v'^2-n^2)^{\frac{3}{2}}+6n^2(v'^2-n^2)^{\frac{1}{2}}-3\pi v'^2n+6v'^2n\sin^{-1}\frac{n}{v'}\right\}s^3\sin^3\phi\,d\theta\,dn\,ds$$

$$=\frac{8}{9}\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v dx\,dy\,dz+\int_{x_1}^a\int_0^{y_1}\int_0^v dx\,dy\,dz\right]dw\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\frac{z'^5}{v'^5}$$

$$\left\{4(v'^2-n^2)^{\frac{3}{2}}+6n^2(v'^2-n^2)^{\frac{1}{2}}-3\pi v'^2n+6v'^2n\sin^{-1}\frac{n}{v'}\right\}(v'^2-n^2)^2\sin^3\phi\,d\theta\,dn$$

$$=\frac{35\pi}{24}\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v v^3z^3\,dx\,dy\,dz+\int_{x_1}^a\int_0^{y_1}\int_0^v v^3z^3\,dx\,dy\,dz\right]dw\int_0^{\frac{1}{2}\pi}z'^2\,d\theta$$

$$=\frac{35\pi^2}{48}\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v v^4z^4\,dx\,dy\,dz+\int_{x_1}^a\int_0^{y_1}\int_0^v v^4z^4\,dx\,dy\,dz\right]dw$$

$$=\frac{7\pi^2}{2^{13}.3}\int_0^a\left[\int_0^{x_1}\int_0^x(2a-x-y)^9\,dx\,dy+\int_{x_1}^a\int_0^{y_1}(2a-x-y)^9\,dx\,dy\right]dw$$

$$=\frac{7\pi^2}{2^{14}.15}\int_0^a\left[\int_0^{x_1}\left\{(2a-x)^{10}-(2a-2x)^{10}\right\}dx\right.$$

$$\left.+\int_{x_1}^a\left\{(2a-x)^{10}-(2a-x-y_1)^{10}\right\}dx\right]dw$$

$$=\frac{\pi^2}{2^{21}.3^2.11.15}\int_0^a\left[14549535a^9w(a^2-w^2)^{\frac{1}{2}}\cos^{-1}\frac{w}{a}+22766336a^{10}w\right.$$

$$-59332833a^9w^2-12713728a^8w^3+149051595a^7w^4$$

$$-6238720a^6w^5-210749154a^5w^6-3296768a^4w^7$$

$$+267525192a^3w^8-508160a^2w^9-246482896aw^{10}$$

$$-64896w^{11}+167483648\frac{w^{12}}{a}-80337280\frac{w^{14}}{a^3}$$

$$\left.-466591744\frac{w^{15}}{a^5}-5132288\frac{w^{18}}{a^7}+458752\frac{w^{20}}{a^9}\right]dw$$

$$=\frac{\pi^2a^{12}}{2^{22}.3^4.5.7.11.13.17.19}(427654482255\pi-5199104301238).$$

Integrating the denominator, we have

$$\pi^3 \int_0^a \left[\int_0^{x_1}\int_0^{x}\int_0^{v} v^3 z^3\,dx\,dy\,dz + \int_{x_1}^{a}\int_0^{y_1}\int_0^{v} v^3 z^3\,dx\,dy\,dz\right] dw$$

$$= \frac{\pi^3}{2^9}\int_0^a \left[\int_0^{x_1}\int_0^{x} (2a-x-y)^7\,dx\,dy + \int_{x_1}^{a}\int_0^{y_1} (2a-x-y)^7\,dx\,dy\right] dw$$

$$= \frac{\pi^3}{2}\int_0^a\left[\int_0^{x_1}\left\{(2a-x)^8-(2a-2x)^8\right\}dx + \int_{x_1}^{a}\left\{(2a-x)^8-(2a-x-y_1)^8\right\}dx\right]dw$$

$$= \frac{\pi^3}{2^{16}.3^2.5.7}\int_0^a\Big[225225a^7w(a^2-w^2)^{\frac{1}{2}}\cos^{-1}\frac{w}{a}+348800a^8w$$
$$-758695a^7w^2-221440a^6w^3+196445a^5w^4$$
$$+284160a^4w^5-1720110a^3w^6-24320a^2w^7$$
$$+1569000aw^8-19120w^9-964960\frac{w^{10}}{a}$$
$$+431600\frac{w^{12}}{a^3}-46720\frac{w^{14}}{a^5}-45760\frac{w^{16}}{a^7}\Big]\,dw$$

$$= \frac{\pi^3 a^{10}}{2^{17}.3^3.5.7.11.17}\,(42117075\pi-255678512).$$

Hence we obtain

$$\Delta = \frac{a^2}{23712\pi}\left\{\frac{427654482255\pi-5199104301238}{42117075\pi-255678512}\right\}.$$

11124. (Professor ZERR.)—A prolate spheroid is divided at random by a plane (1) parallel to the revolving axis, (2) perpendicular to the revolving axis; also an oblate spheroid is divided at random by a plane, (3) parallel to the revolving axis, (4) perpendicular to the revolving axis; and then n points are taken at random within the spheroid; find the chance that all the points lie on the same side of the plane.

Solution.

Let x = the altitude of one of the segments into which the spheroid is divided.

Then (1) the volume of the segment is $\frac{1}{3}\pi\,\frac{b^2}{a^2}\,x^2(3a-x)$, and the

required chance is

$$p = \frac{\int_0^{2a} 2\left[\frac{1}{3}\pi\frac{b^2}{a^2}x^2(3a-x)\right]^n dx}{\int_0^{2a}(\frac{4}{3}\pi ab^2)^n dx} = \frac{1}{2^{2n}a^{3n+1}}\int_0^{2a}(3a-x)^n x^{2n}\,dx$$

$$= \frac{2}{2n+1} + \frac{2^2 n}{(2n+1)(2n+2)} + \frac{2^3 n(n-1)}{(2n+1)(2n+2)(2n+3)} + \dots$$

$$\dots + \frac{(1.2.3.4\dots n)\,2^{n+1}}{(2n+1)(2n+2)\dots(3n+1)}.$$

(2) The volume of segment is $\frac{1}{3}\pi\frac{a}{b}x^2(3b-x)$.

Hence $$p = \frac{\int_0^{2b} 2\left[\frac{1}{3}\pi\frac{a}{b}x^2(3b-x)\right]^n dx}{\int_0^{2b}(\frac{4}{3}\pi ab^2)^n dx}$$

$$= \frac{1}{2^{2n}b^{3n+1}}\int_0^{2b}(3b-x)^n x^{2n}\,dx = \text{same as in (1)}.$$

(3) The volume of the segment is $\frac{1}{3}\pi\frac{a^2}{b^2}(3b-x)x^2$. Hence we get

$$p = \frac{\int_0^{2b} 2\left[\frac{1}{3}\pi\frac{a^2}{b^2}x^2(3b-x)\right]^n dx}{\int_0^{2b}(\frac{4}{3}\pi a^2 b)^n dx}$$

$$= \frac{1}{2^{2n}b^{3n+1}}\int_0^{2b}(3b-x)^n x^{2n}\,dx = \text{same as in (1)}.$$

(4) The volume of the segment is $\frac{1}{3}\pi\frac{b}{a}x^2(3a-x)$. Hence we get

$$p = \frac{\int_0^{2a} 2\left[\frac{1}{3}\pi\frac{b}{a}x^2(3a-x)\right]^n dx}{\int_0^{2a}(\frac{4}{3}\pi a^2 b)^n dx}$$

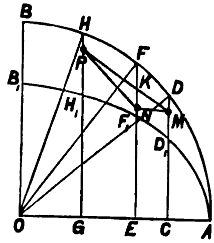
$$= \frac{1}{2^{2n}a^{3n+1}}\int_0^{2a}(3a-x)^n x^{2n}\,dx = \text{same as in (1)};$$

when $n = 2$, $p = \frac{26}{35}$; when $n = 3$, $p = \frac{43}{70}$, &c. Hence we see that p is the same for the four cases, and the same as in a sphere.

11125. (Professor ZERR.)—Find the average area of a triangle formed by joining three points taken at random in the area comprised between the quadrant of an ellipse and its auxiliary circle.

Solution.

Let AOB_1 be the given elliptic quadrant. Let AOB be the quadrant of its auxiliary circle; M, N, P the three random points. Through M, N, P draw CD, EF, GH perpendicular to AO, EF intersecting MP in K.



Let $AO = a$, $OB_1 = b$, $GP = x$,

$CM = y$, $EN = z$, $GH = x'$, $GH_1 = x''$,

$CD = y'$, $CD_1 = y''$, $EF = z'$, $EF_1 = z''$,

$EK = z'''$, $\angle GOH = \theta$, $\angle COD = \phi$,

$$\angle EOF = \psi, \quad v = \frac{1}{\cos\phi - \cos\theta}.$$

Then we have $x' = a\sin\theta$, $x'' = b\sin\theta$, $y' = a\sin\phi$, $y'' = b\sin\phi$,

$$z' = a\sin\psi, \quad z'' = b\sin\psi, \quad z''' = v\left[x(\cos\phi - \cos\psi) + y(\cos\psi - \cos\theta)\right],$$

$$\text{area MNP} = \tfrac{1}{2}a\left[x(\cos\phi - \cos\psi) + y(\cos\psi - \cos\theta) + z(\cos\theta - \cos\phi)\right] = u,$$
when $z < z'''$;

$$\text{area MNP} = \tfrac{1}{2}a\left[x(\cos\psi - \cos\phi) + y(\cos\theta - \cos\psi) + z(\cos\phi - \cos\theta)\right] = u_1,$$
when $z > z'''$.

An element of surface at M is $a\sin\phi\, d\phi\, dy$; at N, $a\sin\psi\, d\psi\, dz$; at P, $a\sin\theta\, d\theta\, dx$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ; of ψ, ϕ and θ; of x, x'' and x'; of y, y'' and y'; of z, z'' and z''', and z''' and z'.

Hence the average area of the triangle is

$$\Delta = \frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_{x''}^{x'}\int_{y''}^{y'}\left\{\int_{z''}^{z'''} u\,dz + \int_{z'''}^{z'} u_1\,dz\right\} a\sin\theta\,d\theta\, a\sin\phi\,d\phi\, a\sin\psi\,d\psi\,dx\,dy}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_{x''}^{x'}\int_{y''}^{y'}\int_{z''}^{z'} a\sin\theta\,d\theta\, a\sin\phi\,d\phi\, a\sin\psi\,d\psi\,dx\,dy\,dz}$$

$$= \frac{384}{\pi^3(a-b)^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_{x''}^{x'}\int_{y''}^{y'}\left\{\int_{z''}^{z'''} u\,dz + \int_{z'''}^{z'} u_1\,dz\right\}$$
$$\times \sin\theta\sin\phi\sin\psi\, d\theta\, d\phi\, d\psi\, dx\, dy$$

$$= \frac{96a}{\pi^3(a-b)^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_{x''}^{x'}\int_{y''}^{y'}\left\{\left[x(\cos\phi - \cos\psi) + y(\cos\psi - \cos\theta) + b\sin\psi(\cos\theta - \cos\phi)\right]^2\right.$$
$$\left. + \left[x(\cos\phi - \cos\psi) + y(\cos\psi - \cos\theta) + a\sin\psi(\cos\theta - \cos\phi)\right]^2\right\}$$
$$\times \sin\theta\sin\phi\sin\psi\, v\, d\theta\, d\phi\, d\psi\, dx\, dy$$

$$\Delta = \frac{32a}{\pi^3 (a-b)^2} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_{\phi}^{\theta} \int_{x''}^{x'} \Big[6x^2 \sin\phi\,(\cos\phi - \cos\psi)^2$$
$$+ 6(a+b)\,x \sin^2\phi\,(\cos\phi-\cos\psi)(\cos\psi-\cos\theta)$$
$$+ 6(a+b)\,x \sin\phi\,\sin\psi\,(\cos\phi-\cos\psi)(\cos\theta-\cos\phi)$$
$$+ 2\,(a^2+ab+b^2)\sin^3\phi\,(\cos\psi-\cos\theta)^2$$
$$+ 3\,(a^2+b^2)\sin\phi\,\sin^2\psi\,(\cos\theta-\cos\phi)^2$$
$$+ 3\,(a+b)^2\sin^2\phi\,\sin\psi\,(\cos\psi-\cos\theta)(\cos\theta-\cos\phi)\Big]$$
$$\times \sin\theta\,\sin\phi\,\sin\psi\,v\,d\theta\,d\phi\,d\psi\,dx$$

$$= \frac{32a\,(a+b)^2}{\pi^3 (a-b)} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_{\phi}^{\theta} \Big[2\sin^3\theta\,\sin\phi\,(\cos\phi-\cos\psi)^2$$
$$+ 2\sin\theta\,\sin^3\phi\,(\cos\psi-\cos\theta)^2$$
$$+ 3\sin^2\theta\,\sin^2\phi\,(\cos\phi-\cos\psi)(\cos\psi-\cos\theta)$$
$$+ 3\sin\theta\,\sin\phi\,\sin^2\psi\,(\cos\theta-\cos\phi)^2$$
$$+ 3\sin^2\theta\,\sin\phi\,\sin\psi\,(\cos\phi-\cos\psi)(\cos\theta-\cos\phi)$$
$$+ 3\sin\theta\,\sin^2\phi\,\sin\psi\,(\cos\psi-\cos\theta)(\cos\theta-\cos\phi)\Big]$$
$$\times \sin\theta\,\sin\phi\,\sin\psi\,v\,d\theta\,d\phi\,d\psi$$

$$- \frac{64a^2 b}{\pi^3 (a-b)} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_{\phi}^{\theta} \Big[\sin^2\theta\,(\cos\phi-\cos\psi)^2 + \sin^2\phi\,(\cos\psi-\cos\theta)^2$$
$$+ 3\sin^2\psi\,(\cos\theta-\cos\phi)^2\Big]\sin^2\theta\,\sin^2\phi\,\sin\psi\,v\,d\theta\,d\phi\,d\psi$$

$$= \frac{16a\,(a+b)}{3\pi^3 (a-b)} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \Big[4\sin^2\theta\cos^2\phi + 4\sin^2\phi\cos^2\phi + 4\sin^2\theta\cos^2\theta$$
$$+ 4\sin^2\phi\cos^2\theta + \sin^2\theta\cos\theta\cos\phi + \sin^2\phi\cos\theta\cos\phi$$
$$- 6\sin\theta\cos\theta\sin\phi\cos\phi + 6\cos^3\theta\cos\phi + 6\cos\theta\cos^3\phi$$
$$+ 12 + 6\cos^2\theta + 6\cos^2\phi - 36\cos\theta\cos\phi - 12\sin\theta\sin\phi$$
$$- 9(\theta-\phi)\sin\theta\cos\phi + 9(\theta-\phi)\sin\phi\cos\theta\Big]\sin^2\theta\,\sin^2\phi\,d\theta\,d\phi$$

$$- \frac{64a^2 b}{3\pi^3 (a-b)} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \Big(4\sin^2\theta\cos^2\theta + \sin^2\theta\cos^2\phi + \sin^2\phi\cos^2\theta$$
$$+ 4\sin^2\phi\cos^2\phi - 5\sin^2\theta\cos\theta\cos\phi - 5\sin^2\phi\cos\theta\cos\phi$$
$$+ 6\cos^2\theta + 6\cos^2\phi - 12\cos\phi\cos\theta\Big)\sin^2\theta\,\sin^2\phi\,d\theta\,d\phi$$

$$= \frac{8a\,(a+b)^2}{9\pi^3 (a-b)} \int_0^{\frac{1}{2}\pi} \Big(69\theta + 36\theta\cos\theta - 12\theta\sin^2\theta - 12\theta\sin^4\theta - 60\sin\theta$$
$$- 45\sin\theta\cos\theta - 10\sin^3\theta\cos\theta + 3\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta$$
$$- \frac{8a^2 b}{9\pi^3 (a-b)} \int_0^{\frac{1}{2}\pi} \Big(105\theta - 30\theta\sin^2\theta - 48\theta\sin^4\theta - 105\sin\theta\cos\theta$$
$$- 40\sin^3\theta\cos\theta + 12\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta\,;$$

thus, finally,

$$\Delta = \frac{a\,(a+b)^2}{\pi\,(a-b)}\left\{\frac{35}{12} + \frac{16}{3\pi} - \frac{131}{3\pi^2}\right\} - \frac{a^2 b}{\pi\,(a-b)}\left\{\frac{35}{12} - \frac{64}{3\pi^2}\right\}.$$

For the area between the semi-ellipse and its auxiliary circle, the

limits of θ are 0 and π, and those of the other variables the same as above. Hence, since the whole number of ways the three points can be taken for the semi-ellipse is eight times the number of ways they can be taken for the quadrant, the average area of the triangle is

$$\Delta_1 = \frac{a(a+b)^2}{9\pi^3(a-b)}\int_0^\pi \Big(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta$$
$$-45\sin\theta\cos\theta-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta$$
$$-\frac{a^2b}{9\pi^3(a-b)}\int_0^\pi\Big(105\theta-30\theta\sin^2\theta-48\theta\sin^4\theta-105\sin\theta\cos\theta$$
$$-40\sin^3\theta\cos\theta+12\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta$$
$$=\frac{a(a+b)^2}{\pi(a-b)}\left\{\frac{35}{24}-\frac{32}{3\pi^2}\right\}-\frac{35}{24}\cdot\frac{a^2b}{\pi(a-b)}.$$

For the area between the whole ellipse and its auxiliary circle, the limits of θ are 0 and 2π, and those of the other variables same as above. Hence, since the whole number of ways the three points can be taken for the whole ellipse is eight times the number of ways for the semi-ellipse, the average area of the triangle is

$$\Delta_2 = \frac{a(a+b)^2}{72\pi^3(a-b)}\int_0^{2\pi}\Big(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta$$
$$-45\sin\theta\cos\theta-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta$$
$$-\frac{a^2b}{72\pi^3(a-b)}\int_0^{2\pi}\Big(105\theta-30\theta\sin^2\theta-48\theta\sin^4\theta-105\sin\theta\cos\theta$$
$$-40\sin^3\theta\cos\theta+12\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta$$
$$=\frac{35}{48}\cdot\frac{a(a+b)^2}{\pi(a-b)}-\frac{35}{48}\cdot\frac{a^2b}{\pi(a-b)}=\frac{35}{48}\cdot\frac{a(a^3-b^3)}{\pi(a-b)^2}.$$

11126. (Professor Zerr.)—Find the average area of a triangle formed by joining three random points in a sphere.

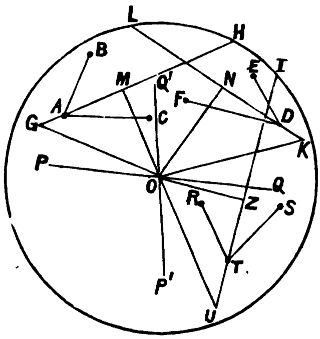
Solution.

Let GH be the diameter of the section of the sphere made by a plane through the three random points A, B, C; M its centre; O the centre of the sphere; OP a line such that AB is parallel to the plane MOP.

Let $\quad OG = r,\quad MA = u,\quad AB = v,\quad AC = w,$

$$\angle GOM = \theta,\quad \angle BAM = \phi,\quad \angle CAM = \psi,\quad \angle MOP = \lambda,$$

and the angle the plane POM makes with a fixed plane through OP $= \rho$.



An element of the sphere is,

at A, $\quad r\sin\theta\, d\theta\, 2\pi u\, du$;

at B, $\quad v^2\, dv\, d\phi\, d\lambda$;

at C, $\sin(\phi+\psi)\sin\lambda w^2\, dw\, d\psi\, d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of u, 0 and $r\sin\theta = u'$, and tripled; of ϕ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\phi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and 2π; of v, 0 and $2u\cos\phi = v'$; of u, 0 and $2u\cos\psi = w'$.

The area of the triangle

$$ABC = \tfrac{1}{2}vw\sin(\phi+\psi).$$

Hence, since the whole number of ways the three points can be taken is $(\frac{4}{3}\pi r^3)^3$, the required average is

$$\Delta = \frac{81}{64\pi^3 r^9}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{v'}\int_0^{w'} \tfrac{1}{2}vw\sin(\phi+\psi)\, r\sin\theta\, d\theta\, 2\pi u\, du$$
$$\times \sin(\phi+\psi)\, d\phi\, d\psi \sin\lambda\, d\lambda\, d\rho\, v^2 dv\, w^2 dw$$

$$= \frac{81}{16\pi^2 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{v'} \sin\theta\, u^5\sin^2(\phi+\psi)\cos^4\psi\sin\lambda\, d\theta\, du\, d\phi$$
$$\times d\psi\, d\lambda\, d\rho\, v^3 dv$$

$$= \frac{81}{4\pi^2 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} \sin\theta\, u^9\sin^2(\phi+\psi)\cos^4\phi\cos^4\psi\sin\lambda$$
$$\times d\theta\, du\, d\phi\, d\psi\, d\lambda\, d\rho$$

$$= \frac{81}{2\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi} \sin\theta\, u^9\sin^2(\phi+\psi)\cos^4\phi\cos^4\psi\sin\lambda\, d\theta\, du\, d\phi\, d\psi\, d\lambda$$

$$= \frac{81}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi} \sin\theta\, u^9\sin^2(\phi+\psi)\cos^4\phi\cos^4\psi\, d\theta\, du\, d\phi\, d\psi$$

$$= \frac{27}{32\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin\theta\, u^9(15\pi\cos^4\phi - 12\pi\cos^6\phi + 30\phi\cos^4\phi - 24\phi\cos^6\phi$$
$$-4\sin\phi\cos^7\phi + 30\sin\phi\cos^5\phi)\, d\theta\, du\, d\phi$$

$$= \frac{27\,.\,15\pi}{32\,.\,4r^8}\int_0^{\frac{1}{2}\pi}\int_0^{u'} \sin\theta\, u^9\, d\theta\, du$$

$$= \frac{81\pi r^2}{256}\int_0^{\frac{1}{2}\pi}\sin^{11}\theta\, d\theta = \frac{9}{77}\pi r^2.$$

11127. (Professor ZERR.) — Find the volume of the tetrahedron formed by joining three points taken at random in a sphere with the centre of the sphere.

Solution.

With the same notation as in the preceding example, we get for the required average, since volume $= \frac{1}{6} vw \sin(\phi+\psi)\, r\cos\theta$,

$$= \frac{81}{64\pi^3 r^9} \int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{v'}\int_0^{w'} \tfrac{1}{6} vw \sin(\phi+\psi)\, r\cos\theta\, r\sin\theta\, d\theta$$
$$\times 2\pi u\, du \sin(\phi+\psi)\, d\phi\, d\psi \sin\lambda\, d\lambda\, d\rho\, v^2 dv\, w^2 dw$$

$$= \frac{27}{16\pi^2 r^7} \int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{v'} \sin\theta\cos\theta\, u^5 \sin^2(\phi+\psi)\cos^4\psi \sin\lambda\, d\theta$$
$$\times du\, d\phi\, d\psi\, d\lambda\, d\rho\, v^3 dv$$

$$= \frac{27}{4\pi^2 r^7} \int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} \sin\theta\cos\theta\, u^9 \sin^2(\phi+\psi)\cos^4\phi\cos^4\psi\sin\lambda$$
$$\times d\theta\, du\, d\phi\, d\psi\, d\lambda\, d\rho$$

$$= \frac{27}{2\pi r^7} \int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi} \sin\theta\cos\theta\, u^9 \sin^2(\phi+\psi)\cos^4\phi\cos^4\psi\sin\lambda$$
$$\times d\theta\, du\, d\phi\, d\psi\, d\lambda$$

$$= \frac{27}{\pi r^7} \int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi} \sin\theta\cos\theta\, u^9 \sin^2(\phi+\psi)\cos^4\phi\cos^4\psi\, d\theta\, du\, d\phi\, d\psi$$

$$= \frac{9}{32\pi r^7} \int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin\theta\cos\theta\, u^9 (15\pi\cos^4\phi - 12\pi\cos^6\phi + 30\phi\cos^4\phi$$
$$- 24\phi\cos^6\phi - 4\sin\phi\cos^7\phi + 30\sin\phi\cos^5\phi)\, d\theta\, du\, d\phi$$

$$= \frac{135\pi}{128 r^7} \int_0^{\frac{1}{2}\pi}\int_0^{u'} \sin\theta\cos\theta\, u^9\, d\theta\, du$$

$$= \frac{27\pi r^3}{256} \int_0^{\frac{1}{2}\pi} \sin^{11}\theta\cos\theta\, d\theta = \frac{9}{1024}\pi r^3.$$

11128. (Professor ZERR.)—Let A, B, C, D, E, F be six random points in a sphere; find the chance that the planes through A, B, C, and D, E, F intersect without the sphere.

Solution.

Let GH, LK be the diameters, through A, D, of the sections of the sphere made by the planes through A, B, C and D, E, F; M, N their centres; O the centre of the sphere; OP a line such that AB is parallel to the plane MOP; OQ a line such that DE is parallel to the plane NOQ.

Let $OG = OK = r$, $MA = a$, $AB = b$, $AC = c$, $ND = u$,

$DE = v$, $DF = w$, $\angle GOM = \theta$, $\angle KON = \phi$, $\angle BAM = \psi$,

$\angle CAM = y$, $\angle MOP = \lambda$,

the angle the plane MOP makes with some fixed plane through $OP = \sigma$,

$\angle EDN = \rho$, $\angle FDN = \delta$, $\angle NOQ = \mu$,

and the dihedral angle MOQN $= \eta$.

An element of the sphere at A is $r \sin\theta\, d\theta \,.\, 2\pi a\, da$; at B, $b^2 db\, d\psi\, d\lambda$; at C, $\sin(\psi + y) \sin\lambda\, c^2 dc\, dy\, d\sigma$; at D, $r \sin\phi\, d\phi \,.\, 2\pi u\, du$; at E, $v^2 dv\, d\rho\, d\mu$; at D, $\sin(\rho + \delta) \sin\mu\, w^2 dw\, d\delta\, d\eta$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$; of a, 0 and $r \sin\theta = a'$, and tripled; of u, 0 and $r \sin\phi = u'$, and tripled; of ψ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of y, $-\psi$ and $+\frac{1}{2}\pi$, and doubled; of λ, 0 and π; of σ, 0 and 2π; of b, 0 and $2a \cos\psi = b'$; of c, 0 and $2a \cos y = c'$; of ρ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of δ, $-\rho$ and $+\frac{1}{2}\pi$, and doubled; of μ, 0 and $\pm(\theta - \phi)$, and $\theta + \phi$ and π (the double sign being taken + when $\theta > \phi$, and − when $\theta \angle \phi$); of w, 0 and 2π; of v, 0 and $2u \cos\rho = v'$; of η, 0 and $2u \cos\delta = w'$.

Since $(\frac{4}{3}\pi r^3)^6$ is the whole number of ways the six points can be taken, the required chance is

$$p = \frac{36 \,.\, 3^6}{4^6 \pi^6 r^{18}} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\psi}^{+\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{b'}\int_0^{c'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\rho}^{+\frac{1}{2}\pi}$$

$$\times \left\{ \int_0^{\pm(\theta-\phi)} \sin\mu\, d\mu + \int_{\theta+\phi}^{\pi} \sin\mu\, d\mu \right\} \int_0^{2\pi}\int_0^{v'}\int_0^{w'}$$

$$\times\, r \sin\theta\, d\theta\, r \sin\phi\, d\phi\, 2\pi a\, da\, 2\pi u\, du\, d\psi \sin(\psi + y)\, dy \sin\lambda\, d\lambda\, d\sigma\, b^2 db\, c^2 dc$$

$$\times\, d\rho \sin(\rho + \delta)\, d\delta\, d\eta\, v^2 dv\, w^2 dw$$

$$= \frac{2187}{32\pi^4 r^{16}} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\psi}^{+\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{b'}\int_0^{c'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\rho}^{+\frac{1}{2}\pi}$$

$$\times \left\{ \int_0^{\pm(\theta-\phi)} \sin\mu\, d\mu + \int_{\theta+\phi}^{\pi} \sin\mu\, d\mu \right\} \int_0^{2\pi}\int_0^{v'}$$

$$\times \sin\theta \sin\phi \sin(\psi + y) \sin(\rho + \delta)\, d\theta\, d\phi\, a\, da\, u^4 du\, d\psi\, dy \sin\lambda\, d\lambda\, d\sigma\, b^2 db$$

$$\times\, c^2 dc\, d\rho \cos^3\delta\, d\delta\, d\eta\, v^2 dv$$

$$= \frac{729}{4\pi^4 r^{16}} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a'}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\psi}^{+\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{b'}\int_0^{c'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\rho}^{+\frac{1}{2}\pi}$$

$$\times \left\{ \int_0^{\pm(\theta-\phi)} \sin\mu\, d\mu + \int_{\theta+\phi}^{\pi} \sin\mu\, d\mu \right\} \int_0^{2\pi}$$

$$\times \sin\theta \sin\phi \sin(\psi + y) \sin(\rho + \delta)\, d\theta\, d\phi\, a\, da\, u^7 du\, d\psi\, dy \sin\lambda\, d\lambda\, d\sigma\, b^2 db\, c^2 dc$$

$$\times \cos^3\rho\, d\rho \cos^3\delta\, d\delta\, d\eta$$

$$= \frac{729}{2\pi^3 r^{16}} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\psi}^{+\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{b'}\int_0^{c'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\rho}^{+\frac{1}{2}\pi}$$

$$\times \left\{ \int_0^{\pm(\theta-\phi)} \sin\mu\, d\mu + \int_{\theta+\phi}^{\pi} \sin\mu\, d\mu \right\} \sin\theta$$

$$\times \sin\phi \sin(\psi + y) \sin(\rho + \delta)\, d\theta\, d\phi\, a\, da\, u^7 du\, d\psi\, dy \sin\lambda\, d\lambda\, d\sigma\, b^2 db\, c^2 dc$$

$$\times \cos^3\rho\, d\rho \cos^3\delta\, d\delta$$

$$= \frac{729}{\pi^3 r^{16}} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\psi}^{+\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{b'}\int_0^{c'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\rho}^{+\frac{1}{2}\pi}$$

$$\times (1 - \sin\theta \sin\phi) \sin\theta \sin\phi \sin(\psi + y) \sin(\rho + \delta)\, d\theta\, d\phi\, a\, da\, u^7 du\, d\psi\, dy$$

$$\times \sin\lambda\, d\lambda\, d\sigma\, b^2 db\, c^2 dc \cos^3\rho\, d\rho \cos^3\delta\, d\delta$$

$$p=\frac{729}{8\pi^3r^{16}}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac12\pi}^{+\frac12\pi}\int_{-\psi}^{+\frac12\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{b'}\int_0^{c'}\int_{-\frac12\pi}^{+\frac12\pi}$$
$$\times\left[3\left(\tfrac12\pi+\rho\right)\sin\rho+2\cos\rho+\sin^2\rho\cos\rho\right](1-\sin\theta\sin\phi)\sin\theta\sin\phi$$
$$\times\sin(\psi+y)\,d\theta\,d\phi\,a\,da\,u^7du\,d\psi\,dy\,\sin\lambda\,d\lambda\,d\sigma\,b^2db\,c^2dc\cos^3\rho\,d\rho$$
$$=\frac{729\,.\,35}{8\,.\,32\pi^2r^{16}}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac12\pi}^{+\frac12\pi}\int_{-\psi}^{+\frac12\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{b'}\int_0^{c'}(1-\sin\theta\sin\phi)\sin\theta\sin\phi$$
$$\times\sin(\psi+y)\,d\theta\,d\phi\,a\,da\,u^7du\,d\psi\,dy\,\sin\lambda\,d\lambda\,d\sigma\,b^2db\,c^2dc$$
$$=\frac{243\,.\,35}{32\pi^2r^{16}}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac12\pi}^{+\frac12\pi}\int_{-\psi}^{+\frac12\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{b'}(1-\sin\theta\sin\phi)\sin\theta\sin\phi$$
$$\times\sin(\psi+y)\,d\theta\,d\phi\,a^4da\,u^7du\,d\psi\,dy\cos^3y\,\sin\lambda\,d\lambda\,d\sigma\,b^2db$$
$$=\frac{9^2.35}{4\pi^2r^{16}}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac12\pi}^{+\frac12\pi}\int_{-\psi}^{+\frac12\pi}\int_0^{\pi}\int_0^{2\pi}(1-\sin\theta\sin\phi)\sin\theta\sin\phi$$
$$\times\sin(\psi+y)\,d\theta\,d\phi\,a^7da\,u^7du\cos^3\psi\,d\psi\cos^3y\,dy\,\sin\lambda\,d\lambda\,d\sigma$$
$$=\frac{9^2.35}{2\pi r^{16}}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac12\pi}^{+\frac12\pi}\int_{-\psi}^{+\frac12\pi}\int_0^{\pi}(1-\sin\theta\sin\phi)\sin\theta\sin\phi\sin(\psi+y)$$
$$\times d\theta\,d\phi\,a^7da\,u^7du\cos^3\psi\,d\psi\cos^3y\,dy\,\sin\lambda\,d\lambda$$
$$=\frac{9^2.35}{\pi r^{16}}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac12\pi}^{+\frac12\pi}\int_{-\psi}^{+\frac12\pi}(1-\sin\theta\sin\phi)\sin\theta\sin\phi\sin(\psi+y)$$
$$\times d\theta\,d\phi\,a^7da\,u^7du\cos^3\psi\,d\psi\cos^3y\,dy$$
$$=\frac{9^2.35}{8\pi r^{16}}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}\int_0^{u'}\int_{-\frac12\pi}^{+\frac12\pi}\left[3\left(\tfrac12\pi+\psi\right)\sin\psi+2\cos\psi+\sin^2\psi\cos\psi\right]$$
$$\times(1-\sin\theta\sin\phi)\sin\theta\sin\phi\,d\theta\,d\phi\,a^7da\,u^7du\cos^3\psi\,d\psi$$
$$=\frac{9^2.35^2}{16^2r^{16}}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}\int_0^{u'}(1-\sin\theta\sin\phi)\sin\theta\sin\phi\,d\theta\,d\phi\,a^7da\,u^7du$$
$$=\frac{9^2.35^2}{8\,.\,16^2r^8}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\int_0^{a'}(1-\sin\theta\sin\phi)\sin\theta\sin^9\phi\,d\theta\,d\phi\,a^7da$$
$$=\frac{9^2.35^2}{8^2.16^2}\int_0^{\frac12\pi}\int_0^{\frac12\pi}(\sin^9\theta\sin^9\phi-\sin^{10}\theta\sin^{10}\phi)\,d\theta\,d\phi$$
$$=\frac{9^2.35^2}{8^2.16^2}\int_0^{\frac12\pi}\left(\frac{8\,.\,16}{9\,.\,35}\sin^9\theta-\frac{63\pi}{16\,.\,32}\sin^{10}\theta\right)d\theta=1-\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2.$$

Hence the chance that the planes intersect within the sphere is

$$p_1=1-p=\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2.$$

11129. (Professor ZERR.)—If a third plane be passed through the three random points R, T, S, prove that (1) the chance that the plane

through D, E, F is cut by both the others within the sphere is

$$\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2\times\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2=\left(\frac{63}{64}\right)^8\left(\frac{5\pi}{16}\right)^4;$$

(2) the chance that the same plane is cut by one, and not by the other within the sphere is $\left\{1-\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2\right\}\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2$;

and (3) the chance p_2 of the plane through D, E, F being cut within the sphere is

$$p_2=2\left\{1-\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2\right\}\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2+\left(\frac{63}{64}\right)^8\left(\frac{5\pi}{16}\right)^4;$$

p_3, the chance of its not being cut within the sphere, is

$$p_3=\left\{1-\left(\frac{63}{64}\right)^4\left(\frac{5\pi}{16}\right)^2\right\}^2.$$

Solution.

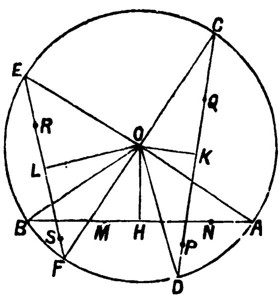
These results follow from the solution of the previous Question, 11128, and we have $p_2+p_3=1$, as it should be.

11130. (Professor ZERR.)—A chord is drawn at random across a circle, and two points are taken at random within the circle; find the chance that both points lie on the same side of the random chord.

Solution.

Let $\angle AOH=\theta$,

and AB be the random chord.



Area sector ADFB is

$$r^2(\theta-\sin\theta\cos\theta).$$

Hence the required chance is

$$p=\frac{\int_0^\pi 2\left[r^2(\theta-\sin\theta\cos\theta)\right]^2 r\sin\theta\,d\theta}{\int_0^\pi(\pi r^2)^2 r\sin\theta\,d\theta}$$

$$=\frac{1}{\pi^2}\int_0^\pi(\theta-\sin\theta\cos\theta)^2\sin\theta\,d\theta$$

$$=1-\frac{128}{45\pi^2}.$$

The chance that both points do not lie on the same side of the random chord is
$$p_1 = 1 - p = \frac{128}{45\pi^2}.$$

11131. (Professor Zerr.)—A line crosses a circle at random; find the chance that a point taken at random within the circle shall be at a distance from this line greater than the radius of the circle.

Solution.

With the same notation as above, we have

$$p = \frac{\int_0^{\frac{1}{2}\pi} r^2 (\theta - \sin\theta\cos\theta)\, r\sin\theta\, d\theta}{\int_0^{\frac{1}{2}\pi} \pi r^2 r \sin\theta\, d\theta} = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} (\theta - \sin\theta\cos\theta)\sin\theta\, d\theta = \frac{2}{3\pi}.$$

The chance that the point is distant by less than the radius is

$$p_1 = 1 - p = 1 - \frac{2}{3\pi}.$$

11132. (Professor Zerr.)—Two chords are drawn, each through two random points in the surface of a given circle; find the chance that they will not intersect.

Solutions.

1. Let M, N be the first two points, AB the chord through them, P, Q the second two points, CD the chord through them. Draw OH, OK perpendicular to AB, CD.

Let $\quad OA = r, \quad AM = a, \quad MN = b, \quad CP = x, \quad PQ = y,$

$$\angle AOH = \theta, \quad \angle COK = \phi, \quad \angle KOH = \psi,$$

and ρ the angle AB makes with some fixed line.

An element of the circle at M is $r\sin\theta\, d\theta\, da$; at N it is $b\,db\,d\rho$; at P, $r\sin\phi\, d\phi\, dx$; at Q, $y\,dy\,d\psi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ, and doubled; of ψ, 0 and $\theta - \phi$, and $\theta + \phi$ and π, and doubled; of ρ, 0 and 2π; of a, 0 and $2r\sin\theta = a'$; of b, 0 and a, and doubled; of x, 0 and $2r\sin\phi = x'$; of y, 0 and x, and doubled.

Since $\pi^4 r^8$ is the whole number of ways the four points can be taken, the required chance is

$$p = \frac{16}{\pi^4 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\int_0^{\theta-\phi}d\psi+\int_{\theta+\phi}^{\pi}d\psi\right\}\int_0^{2\pi}\int_0^{a'}\int_0^{a}\int_0^{x'}\int_0^{x} r\sin\theta\, d\theta\, r\sin\phi\, d\phi\, d\rho \times da\, b\, db\, dx\, y\, dy$$

$$= \frac{8}{\pi^4 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\int_0^{\theta-\phi}d\psi+\int_{\theta+\phi}^{\pi}d\psi\right\}\int_0^{2\pi}\int_0^{a'}\int_0^{a}\int_0^{x'}\sin\theta\sin\phi\, d\theta\, d\phi\, d\rho\, da\, b\, db\, x^2 dx$$

$$= \frac{64}{3\pi^4 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\int_0^{\theta-\phi}d\psi+\int_{\theta+\phi}^{\pi}d\psi\right\}\int_0^{2\pi}\int_0^{a'}\int_0^{a}\sin\theta\sin^4\phi\, d\theta\, d\phi\, d\rho\, da\, b\, db$$

$$= \frac{32}{3\pi^4 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\int_0^{\theta-\phi}d\psi+\int_{\theta+\phi}^{\pi}d\psi\right\}\int_0^{2\pi}\int_0^{a'}\sin\theta\sin^4\phi\, d\theta\, d\phi\, d\rho\, a^2 da$$

$$= \frac{256}{9\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\int_0^{\theta-\phi}d\psi+\int_{\theta+\phi}^{\pi}d\psi\right\}\int_0^{2\pi}\sin^4\theta\sin^4\phi\, d\theta\, d\phi\, d\rho$$

$$= \frac{512}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\int_0^{\theta-\phi}d\psi+\int_{\theta+\phi}^{\pi}d\psi\right\}\sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{1024}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}(\tfrac{1}{2}\pi-\phi)\sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{64}{9\pi^3}\int_0^{\frac{1}{2}\pi}\Big(3\pi\theta-2\pi\sin^3\theta\cos\theta-3\pi\sin\theta\cos\theta-3\theta^2+4\theta\sin^3\theta\cos\theta + 6\theta\sin\theta\cos\theta-\sin^4\theta-3\sin^2\theta\Big)\sin^4\theta\, d\theta$$

$$= \frac{2}{3}-\frac{245}{72\pi^2}.$$

2. *Otherwise*:—Let $\angle$ AOC $= \psi$.
The limits of θ are 0 and π; of ψ, 2ϕ and 2θ; of ϕ, 0 and θ; and for the other variables same as in first solution. Hence we have

$$p = \frac{4}{\pi^4 r^8}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\int_0^{a'}\int_0^{a}\int_0^{x'}\int_0^{x} r\sin\theta\, d\theta\, r\sin\phi\, d\phi\, d\psi\, d\mu\, da\, b\, db\, dx\, y\, dy$$

$$= \frac{2}{\pi^4 r^6}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\int_0^{a'}\int_0^{a}\int_0^{x'}\sin\theta\sin\phi\, d\theta\, d\phi\, d\psi\, d\mu\, da\, b\, db\, x^2 dx$$

$$= \frac{16}{3\pi^4 r^3}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\int_0^{a'}\int_0^{a}\sin\theta\sin^4\phi\, d\theta\, d\phi\, d\psi\, d\mu\, da\, b\, db$$

$$= \frac{8}{3\pi^4 r^3}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\int_0^{a'}\sin\theta\sin^4\phi\, d\theta\, d\phi\, d\psi\, d\mu\, a^2 da$$

$$= \frac{64}{9\pi^4}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\sin^4\theta\sin^4\phi\, d\theta\, d\phi\, d\psi\, d\mu$$

$$= \frac{128}{9\pi^3}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\sin^4\theta\sin^4\phi\, d\theta\, d\phi\, d\psi = \frac{256}{9\pi^3}\int_0^{\pi}\int_0^{\theta}(\theta-\phi)\sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{16}{9\pi^3}\int_0^{\pi}(3\theta^2-3\sin^2\theta-\sin^4\theta)\sin^4\theta\, d\theta = \frac{2}{3}-\frac{245}{72\pi^2}.$$

The chance that the chords will intersect is

$$1-p = \frac{1}{3}+\frac{245}{72\pi^2}.$$

11133. (Professor ZERR.)—If EF be the chord through the two random points R, S; prove that (1) the chance that both the chords EF and CD intersect AB is $p_1 = \left\{\frac{1}{3} + \frac{245}{72\pi^2}\right\}^2$;

(2) the chance that CD or EF intersects and EF or CD does not intersect is $p_2 = \left\{\frac{2}{3} - \frac{245}{72\pi^2}\right\}\left\{\frac{1}{3} + \frac{245}{72\pi^2}\right\}$;

(3) the chance then that AB will be cut is

$$p_3 = 2\left(\frac{2}{3} - \frac{245}{72\pi^2}\right)\left(\frac{1}{3} + \frac{245}{72\pi^2}\right) + \left(\frac{1}{3} + \frac{245}{72\pi^2}\right)^2;$$

and (4) the chance that AB will not be cut is

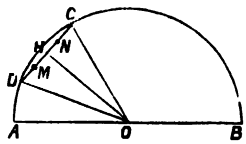
$$p_4 = \left(\frac{2}{3} - \frac{245}{72\pi^2}\right)^2.$$

Solution.

This follows from the solution of Quest. 11132; and we have $p_3 + p_4 = 1$, as it should be.

11134. (Professor ZERR.)—Two points are taken at random within a semicircle; find the chance that the chord through them intersects the arc in two points.

Solution.



Let ACB be the semicircle, O its centre, M, N the two random points, CD the chord through them.

Draw OH perpendicular to CD.

Let $OA = r$, $CM = x$, $mn = y$,

$\angle AOH = \theta$, $\angle COH = \phi$.

An element of surface at M is $r \sin\phi \, d\phi \, dx$, as at N it is $d\theta \, y \, dy$.

The limits of θ are 0 and $\frac{1}{2}\pi$, and doubled; of ϕ, 0 and θ; of x, 0 and $2r\sin\phi = x'$; of y, 0 and x, and doubled.

Since $\frac{1}{4}\pi^2 r^4$ is the whole number of ways the two points can be taken, the required chance is

$$p = \frac{16}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'}\int_0^{x} r\sin\phi \, d\theta \, d\phi \, dx \, y \, dy$$

$$= \frac{8}{\pi^2 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'} \sin\phi \, d\theta \, d\phi \, x^2 dx = \frac{64}{3\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta} \sin^4\phi \, d\theta \, d\phi$$

$$= \frac{8}{3\pi^2}\int_0^{\frac{1}{2}\pi} (3\theta - 2\sin^3\theta\cos\theta - 3\sin\theta\cos\theta)\, d\theta = 1 - \frac{16}{3\pi^2}.$$

11135. (Professor ZERR.)—Required the chance that the chord drawn through two random points inside a quadrant of a circle cuts the arc at two points.

Solution.

With the same notation as in the previous example, we have the limits of θ, 0 and $\frac{1}{2}\pi$; and of the other variables same as in previous example.

Since $\frac{1}{16}\pi^2 r^4$ is the whole number of ways the two points can be taken, the required chance is

$$p = \frac{64}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'}\int_0^{x} r\sin\phi\, d\theta\, d\phi\, dx\, y\, dy$$

$$= \frac{32}{\pi^2 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{r'} \sin\phi\, d\theta\, d\phi\, x^2 dx = \frac{256}{3\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\sin^4\phi\, d\theta\, d\phi$$

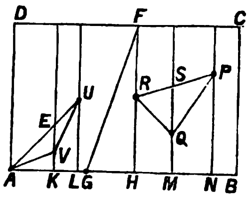
$$= \frac{32}{3\pi^2}\int_0^{\frac{1}{2}\pi}(3\theta - 2\sin^3\theta\cos\theta - 3\sin\theta\cos\theta)\, d\theta = 1 - \frac{28}{3\pi^2}.$$

The chance that the chord intersects in one point is $\frac{8}{\pi^2}$,

The chance that the chord does not intersect the arc is $\frac{4}{3\pi^2}$.

11136. (Professor ZERR.)—Find the mean distance between two points on opposite sides of a rectangle.

Solution.



Let ABCD be the rectangle, F, G the two points on opposite sides.

Let $AB = a$, $BC = b$,

$AH = x$, $AG = x'$.

Then $FG = \{b^2 + (x - x')^2\}^{\frac{1}{2}}$.

The limits of x are 0 and a; of x', 0 and x.

Hence the required average is

$$\Delta = \frac{\int_0^a\int_0^x \{b^2 + (x-x')^2\}^{\frac{1}{2}}\, dx\, dx'}{\int_0^a\int_0^x dx\, dx'} = \frac{2}{a^2}\int_0^a\int_0^x \{b^2 + (x-x')^2\}^{\frac{1}{2}}\, dx\, dx'$$

$$= \frac{1}{a^2}\int_0^a \left\{x(b^2+x^2)^{\frac{1}{2}} + b^2\log\left[x + (b^2+x^2)^{\frac{1}{2}}\right] - b^2\log b\right\} dx$$

$$= \frac{1}{3a^2}(a^2+b^2)^{\frac{3}{2}} + \frac{b^2}{a}\log\{a + (a^2+b^2)^{\frac{1}{2}}\} - \frac{b^2}{a}\log b - \frac{1}{a^2}(a^2+b^2)^{\frac{1}{2}} - \frac{b^3}{3a^2} + \frac{b}{a^2}.$$

For two points of CB, AD, we get, by writing a for b, and b for a,

$$\Delta_1 = \frac{1}{3b^2}(a^2+b^2)^{\frac{3}{2}} + \frac{a^2}{b}\log\{b+(a^2+b^2)^{\frac{1}{2}}\} - \frac{a^2}{b}\log a - \frac{1}{b^2}(a^2+b^2)^{\frac{1}{2}} - \frac{a^3}{3b^2} + \frac{a}{b^2}.$$

If $a = b$, $\quad \Delta = \dfrac{2a\sqrt{2}}{3} + a\log(1+\sqrt{2}) - \dfrac{1}{a}\sqrt{2} - \dfrac{1}{3}a + \dfrac{1}{a}$.

If $a = 1$, $\quad \Delta = \dfrac{2-\sqrt{2}}{3} + \log(1+\sqrt{2})$.

11137. (Professor ZERR.)—Find the average area of a triangle formed by joining two random points in a rectangle to one of its angles.

Solution.

Let U, V be the random points, and let LU, KE be drawn through U, V perpendicular to AB.

Put $\quad$ AL $= x$, $\quad$ AK $= w$, $\quad$ LU $= y$, $\quad$ KV $= z$, $\quad$ KE $= z'$.

Then we have $\qquad z' = \dfrac{wy}{x}$; also

$$\text{Area AUV} = \tfrac{1}{2}(wy - xz) = u, \quad \text{when } z < z';$$
$$\text{Area AUV} = \tfrac{1}{2}(xz - wy) = u_1, \quad \text{when } z > z'.$$

Limits of x are 0 and a; of w, 0 and x; of y, 0 and b; of z, 0 and z', and z' and b.

Hence the required average area is

$$\Delta = \frac{\int_0^a\int_0^x\int_0^b\left\{\int_0^{z'} u\,dz + \int_{z'}^b u_1\,dz\right\}dx\,dw\,dy}{\int_0^a\int_0^x\int_0^b\int_0^b dx\,dw\,dy\,dz}$$

$$= \frac{2}{a^2b^2}\int_0^a\int_0^x\int_0^b\left\{\int_0^{z'} u\,dz + \int_{z'}^b u_1\,dz\right\}dx\,dw\,dy$$

$$= \frac{1}{2a^2b^2}\int_0^a\int_0^x\int_0^b\left(\frac{2w^2y^2}{x} + b^2x - 2bwy\right)dx\,dw\,dy$$

$$= \frac{b}{6a^2}\int_0^a\int_0^x\left(\frac{2w^2}{x} + 3x - 3w\right)dx\,dw = \frac{b}{36a^2}\int_0^a 13x^2dx = \frac{13ab}{108}.$$

If $a = b$, we have $\qquad \Delta = \dfrac{13}{108}a^2$.

11138. (Professor ZERR.)—**Find the average area of a triangle formed by joining three random points in a rectangle.**

Solution.

Let P, Q, R be the three random points in the rectangle ABCD. Draw, through P, Q, R, NP, MS, HR perpendicular to AB. Let

$$AB = a, \quad BC = b, \quad An = u, \quad AH = v. \quad AM = w, \quad HR = x,$$
$$NP = y, \quad MQ = z, \quad MS = z'.$$

Then we have

$$\text{area PQR} = \tfrac{1}{2}\left[x(u-w) + y(w-v) + z(v-u)\right] = A, \quad \text{when } z < z';$$
$$\text{area PQR} = \tfrac{1}{2}\left[x(w-u) + y(v-w) + z(u-v)\right] = A_1, \quad \text{when } z > z';$$
$$z' = \frac{x(u-w) + y(w-v)}{u-v}.$$

The limits of u are 0 and a; of v, 0 and u; of w, v and u; of x, 0 and b; of y, 0 and b; of z, 0 and z', and z' and b.

Hence the required average area is

$$\Delta = \frac{\int_0^a\int_0^u\int_v^u\int_0^b\int_0^b \left\{\int_0^{z'} A\,dz + \int_{z'}^b A_1\,dz\right\} du\,dv\,dw\,dx\,dy}{\int_0^a\int_0^u\int_v^u\int_0^b\int_0^b\int_0^b du\,dv\,dw\,dx\,dy\,dz}$$

$$= \frac{6}{a^3b^3}\int_0^a\int_0^u\int_v^u\int_0^b\int_0^b \left\{\int_0^{z'} A\,dz + \int_{z'}^b A_1\,dz\right\} du\,dv\,dw\,dx\,dy$$

$$= \frac{3}{2a^3b^3}\int_0^a\int_0^u\int_v^u\int_0^b\int_0^b \left\{\left[x(u-w) + y(w-v)\right]^2 + \left[x(u-w) + y(w-v) + b(v-u)\right]^2\right\} du\,dv\,dw\,\frac{1}{u-v}\,dx\,dy$$

$$= \frac{1}{2a^3b^2}\int_0^a\int_0^u\int_v^u\int_0^b \left[6x^2(u-w)^2 + 6bx(u-w)(w-v) + 6bx(u-w)(v-u) + 2b^2(w-v)^2 + 3b^2(v-u)^2 + 3b^2(w-v)(v-u)\right] \times du\,dv\,\frac{1}{u-v}\,dw\,dx$$

$$= \frac{b}{2a^3}\int_0^a\int_0^u\int_v^u (2u^2 + 2v^2 + w^2 - 3uv - uw - vw)\,du\,dv\,\frac{1}{u-v}\,dw$$

$$= \frac{11b}{12a^3}\int_0^a\int_0^u (u-v)^2\,du\,dv = \frac{11b}{36a^3}\int_0^a u^3\,du = \frac{11ab}{144}.$$

If $a = b$, we have $\Delta = \frac{11}{144}a^2.$

The chance that a fourth random point in ABCD falls on PQR is

$$\text{PQR} \div \text{ABCD} = \frac{11}{144}.$$

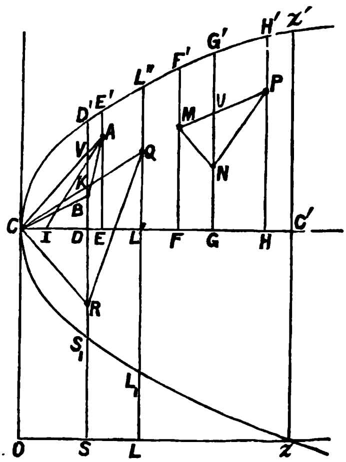
11139. (Professor Zerr.)—Prove that (1) the chance then that four random points in a rectangle form a re-entrant quadrilateral is $\frac{11}{36}$, and (2) the chance of a convex quadrilateral is $\frac{25}{36}$.

Solution.

This follows at once from the foregoing Solution of Quest. 11138.

11140. (Professor Zerr.)—Two points are taken at random in the surface above the transverse axis of a parabola whose base is $2b$ and altitude h; find the average area of the triangle formed by joining the two points to the vertex.

Solution.



Let CC′Z′ be the given surface, A, B the random points.

Let $CC' = h$, $C'Z' = b$,

$$CE = x, \quad CD = y, \quad EA = z,$$

$$DB = w, \quad DV = w'',$$

$$DD' = (py)^{\frac{1}{2}} = y',$$

$$EE' = (px)^{\frac{1}{2}} = x';$$

$$\text{area ABC} = \tfrac{1}{2}(zy - wx) = u, \text{ when } w < w'';$$

$$\text{area ABC} = \tfrac{1}{2}(wx - zy) = u_1, \text{ when } w > w'';$$

$$w'' = \frac{yz}{x}.$$

The limits of x are 0 and h; of y, 0 and x; of z, 0 and x'; of w, 0 and w'', and w'' and y'.

Hence the required average is

$$\Delta = \frac{\int_0^h\int_0^x\int_0^{x'}\left\{\int_0^{w''} u\,dw + \int_{w''}^{y'} u_1\,dw\right\}dx\,dy\,dz}{\int_0^h\int_0^x\int_0^{x'}\int_0^{y'} dx\,dy\,dz\,dw}$$

$$= \frac{9}{2b^2h^2}\int_0^h\int_0^x\int_0^{x'}\left\{\int_0^{w''} u\,dw + \int_{w''}^{y'} u_1\,dw\right\}dx\,dy\,dz$$

$$\Delta = \frac{9}{8b^2h^2}\int_0^h\int_0^x\int_0^{z'}\left(\frac{2y^2z^2}{x}+pxy-2yz\,(py)^{\frac{1}{2}}\right)dx\,dy\,dz$$

$$= \frac{3p^{\frac{1}{2}}}{8b^2h^2}\int_0^h\int_0^x(2y^2x^{\frac{1}{2}}+3x^{\frac{3}{2}}y-3xy^{\frac{3}{2}})\,dx\,dy = \frac{29p^{\frac{1}{2}}}{80b^2h^2}\int_0^h x^{\frac{7}{2}}\,dx = \frac{29bh}{360}.$$

11141. (Professor ZERR.)—Find the average area of a triangle formed by joining two random points anywhere in the surface of a parabola to its vertex.

Solution.

Let Q, R be the random points. Let $CC' = h$, $CO = b$, OZ being parallel to CC′, and CO parallel to C′Z,

$$OL = x,\quad OS = y,\quad LQ = z,$$

$$SR = w,\quad SK = w'',\quad SD' = w',\quad SS_1 = w''',$$

$$LL'' = z',\quad LL_1 = z''.$$

Then we have

$$w' = b+(py)^{\frac{1}{2}},\quad w''' = b-(py)^{\frac{1}{2}},\quad w'' = \frac{bx-by+yz}{x},$$

$$z' = b+(px)^{\frac{1}{2}},\quad z'' = b-(px)^{\frac{1}{2}},$$

$$\text{area CQR} = \tfrac{1}{2}\left[x\,(b-w)+y\,(z-b)\right] = u,\quad \text{when } w<w'';$$

$$\text{area CQR} = \tfrac{1}{2}\left[x\,(w-b)+y\,(b-z)\right] = u_1,\quad \text{when } w>w''.$$

The limits of x are 0 and h; of y, 0 and x; of z, z'' and z'; of w, w'' and w''', and w' and w''.

And the required average area is

$$= \frac{\int_0^h\int_0^x\int_{z''}^{z'}\left\{\int_{w'''}^{w''}u\,dw+\int_{w''}^{w'}u_1\,dw\right\}dx\,dy\,dz}{\int_0^h\int_0^x\int_{z''}^{z'}\int_{w'''}^{w'}dx\,dy\,dz\,dw}$$

$$= \frac{9}{8b^2h^2}\int_0^h\int_0^x\int_{z''}^{z'}\left\{\int_{w'''}^{w''}u\,dw+\int_{w''}^{w'}u_1\,dw\right\}dx\,dy\,dz$$

$$= \frac{9}{16b^2h}\int_0^h\int_0^x\int_{z''}^{z'}\left\{\frac{(bx-by+yz)^2}{x}-2b\,(bx-by+yz)+x\,(b^2+py)\right\}dx\,dy\,dz$$

$$= \frac{3p^{\frac{1}{2}}}{16b^2h^2}\int_0^h\int_0^x(6yx^{\frac{3}{2}}+2y^2x^{\frac{1}{2}})\,dx\,dy = \frac{11p^{\frac{1}{2}}}{16b^2h^2}\int_0^h x^{\frac{7}{2}}\,dx = \frac{11}{72}bh.$$

11142. (Professor ZERR.)—**Find the average area of a triangle formed by joining two random points, one on each side of the transverse axis, to the vertex of a parabola.**

Solution.

Using the same triangle and the same notation as in the last example, we have $\quad$ area CQR $= \frac{1}{2}\left[x(b-w)+y(z-b)\right] = u.$

The limits of x are 0 and h; of y, 0 and x; of z, b and $b+(px)^{\frac{1}{2}} = z'$, of w, $b-(py)^{\frac{1}{2}} = w'$, and b.

Hence the required average is

$$\Delta = \frac{\int_0^h\int_0^x\int_b^{z'}\int_{w'}^b u\,dx\,dy\,dz\,dw}{\int_0^h\int_0^x\int_b^{z'}\int_{w'}^b dx\,dy\,dz\,dw} = \frac{9}{2b^2h^2}\int_0^h\int_0^x\int_b^{z'}\int_{w'}^b u\,dx\,dy\,dz\,dw$$

$$= \frac{9}{8b^2h^2}\int_0^h\int_0^x\int_b^{z'}\left(pxy+2y^{\frac{3}{2}}zp^{\frac{1}{2}}-2by^{\frac{3}{2}}p^{\frac{1}{2}}\right)dx\,dy\,dz$$

$$= \frac{9p^{\frac{3}{2}}}{8b^2h^2}\int_0^h\int_0^x\left(x^{\frac{3}{2}}y+xy^{\frac{3}{2}}\right)dx\,dy = \frac{81p^{\frac{3}{2}}}{80b^2h^2}\int_0^h x\,dx = \frac{9bh}{40}.$$

11143. (Professor ZERR.)—Prove that the chance of (1) a re-entrant, (2) a convex, quadrilateral being formed by joining three random points in a parabola to the vertex, is for (1) $\frac{11}{32}$, for (2) $\frac{21}{32}$.

Solution.

This follows at once from the foregoing solution of Question 11142.

11144. (Professor ZERR.)—Find the average area of a triangle formed by joining three points taken at random in the surface, above the transverse axis, of a parabola.

Solution.

Let M, N, P be the random points; through M, N, P draw FF′, GG′, HH′ perpendicular to CC′, GG′, meeting MP in U.

Let $\quad CC' = h,\quad C'Z' = b,\quad CH = u,\quad CF = v,\quad CG = w,$

$$HP = x,\quad FM = y,\quad GN = z,\quad GU = z'',$$

$$FF' = y',\quad GG' = z',\quad HH' = x'.$$

Then we have

$$\text{Area MNP} = \tfrac{1}{2}\left[x(w-v)+y(u-w)+z(v-u)\right] = \mathrm{A}, \quad \text{when } z<z'';$$

$$\text{Area MNP} = \tfrac{1}{2}\left[x(v-w)+y(w-u)+z(u-v)\right] = \mathrm{A}_1, \quad \text{when } z>z'';$$

$$x' = (pu)^{\frac{1}{2}}, \quad y' = (pv)^{\frac{1}{2}}, \quad z' = (pw)^{\frac{1}{2}}, \quad z'' = \frac{x(w-v)+y(u-w)}{u-v}.$$

The limits of u are 0 and h; of v, 0 and u; of w, v and u; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'', and z'' and z'.

Hence the required average area is

$$\Delta = \frac{\int_0^h\int_0^u\int_v^u\int_0^{x'}\int_0^{y'}\left\{\int_0^{z''}\mathrm{A}\,dz+\int_{z''}^{z'}\mathrm{A}_1\,dz\right\}du\,dv\,dw\,dx\,dy}{\int_0^h\int_0^u\int_v^u\int_0^{x'}\int_0^{y'}\int_0^{z'}du\,dv\,dw\,dx\,dy\,dz}$$

$$= \frac{81}{4b^3h^3}\int_0^h\int_0^u\int_v^u\int_0^{x'}\int_0^{y'}\left\{\int_0^{z''}\mathrm{A}\,dz+\int_{z''}^{z'}\mathrm{A}_1\,dz\right\}du\,dv\,dw\,dx\,dy$$

$$= \frac{81}{16b^3h^3}\int_0^h\int_0^u\int_v^u\int_0^{x'}\int_0^{y'}\left\{\frac{\left[x(w-v)+y(u-w)\right]^2}{u-v}\right.$$
$$\left.+2(pw)^{\frac{1}{2}}\left[x(v-w)+y(w-u)\right]+pw(u-v)\right\}du\,dv\,dw\,dx\,dy$$

$$= \frac{27}{16b^3h^3}\int_0^h\int_0^u\int_v^u\int_0^{x'}\left[\frac{6x^2(pv)^{\frac{1}{2}}(w-v)^2}{u-v}+\frac{6xpv(w-v)(u-w)}{u-v}\right.$$
$$+\frac{2(pv)^{\frac{3}{2}}(u-w)^2}{u-v}+6xp(wv)^{\frac{1}{2}}(v-w)+3p^{\frac{3}{2}}vw^{\frac{1}{2}}(w-u)$$
$$\left.+3p^{\frac{3}{2}}wv^{\frac{1}{2}}(u-v)\right]du\,dv\,dw\,dx$$

$$= \frac{27p^2}{16b^3h}\int_0^h\int_0^u\int_v^u\left[\frac{2u^{\frac{3}{2}}v^{\frac{1}{2}}(w-v)^2}{u-v}+\frac{3uv(w-v)(u-w)}{u-v}+\frac{2u^{\frac{1}{2}}v^{\frac{3}{2}}(u-w)^2}{u-v}\right.$$
$$\left.+3uw^{\frac{1}{2}}v^{\frac{1}{2}}(v-w)+3u^{\frac{1}{2}}vw^{\frac{1}{2}}(w-u)+3u^{\frac{1}{2}}v^{\frac{1}{2}}w(u-v)\right]du\,dv\,dw$$

$$= \frac{9p^2}{160b^3h^3}\int_0^h\int_0^u\left(29u^{\frac{7}{2}}v^{\frac{1}{2}}+29u^{\frac{1}{2}}v^{\frac{7}{2}}-5u^{\frac{5}{2}}v^{\frac{3}{2}}-5u^{\frac{3}{2}}v^{\frac{5}{2}}-30u^2v^2\right.$$
$$\left.-9u^3v-9uv^3\right)du\,dv$$

$$\doteq \frac{1411p^2}{4480b^3h^3}\int_0^h u^5\,du = \frac{1411bh}{26880}.$$

11145. (Professor ZERR.)—Find the mean distance of a point within a parabola from the vertex.

Solution.

Let Q be the point, CO $= b$, OZ $= h$, OL $= x$, LQ $= y$. Then we have

$$CQ = \{x^2 + (y-b)^2\}^{\frac{1}{2}}.$$

The limits of x are 0 and h; of y, $b + (px)^{\frac{1}{2}} = y'$, and $b - (px)^{\frac{1}{2}} = y''$; hence the required mean is

$$M = \frac{\int_0^h \int_{y''}^{y'} \{x^2 + (y-b)^2\}^{\frac{1}{2}}\, dx\, dy}{\int_0^h \int_{y''}^{y'} dx\, dy} = \frac{3}{4bh} \int_0^h \int_{y''}^{y'} \{x^2 + (y-b)^2\}^{\frac{1}{2}}\, dx\, dy$$

$$= \frac{3}{4bh} \int_0^h \left[p^{\frac{1}{2}} x\, (x+p)^{\frac{1}{2}} + x^2 \log \left\{ \frac{p^{\frac{1}{2}} + (x+p)}{x^{\frac{1}{2}}} \right\} \right] dx$$

$$= \frac{1}{4bh^4} \left\{ \left(\frac{7}{5} bh^4 + \frac{2}{15} b^3 h^2 - \frac{4}{15} b^5 \right) (b^2 + h^2)^{\frac{1}{2}} + \frac{4}{15} b^6 + h^6 \log \left[\frac{b + (b^2 + h^2)^{\frac{1}{2}}}{h} \right] \right\}.$$

11146. (Professor Zerr.)—Find the mean distance of a point within a parabola from the focus.

Solution.

Let I be the focus, A the point,

CO $= b$, OL $= h$, CI $= \frac{1}{4}p = a$, CE $= x$, AE $= y - b$.

Then $$IA = \{(x-a)^2 + (y-b)^2\}^{\frac{1}{2}}.$$

The limits of x are 0 and h; of y, $b + 2\,(ax)^{\frac{1}{2}} = y'$ and $b - 2\,(ax)^{\frac{1}{2}} = y''$.

The required mean is

$$M = \frac{\int_0^h \int_{y''}^{y'} \{(x-a)^2 + (y-b)^2\}^{\frac{1}{2}}\, dx\, dy}{\int_0^h \int_{y''}^{y'} dx\, dy} = \frac{3}{4bh} \int_0^h \int_{y''}^{y'} \{(x-a)^2 + (y-b)^2\}^{\frac{1}{2}}\, dx\, dy$$

$$= \frac{3}{4bh} \int_0^h \left\{ 2\,(ax)^{\frac{1}{2}}\,(x+a) + (x-a)^2 \log \left(\frac{x^{\frac{1}{2}} + a^{\frac{1}{2}}}{x^{\frac{1}{2}} - a^{\frac{1}{2}}} \right) \right\} dx$$

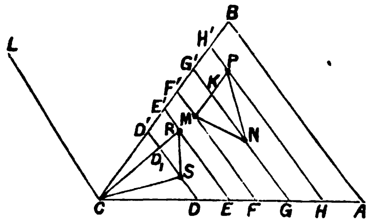
$$= \frac{1}{4bh} \left[\tfrac{7}{5} bh^2 + \tfrac{1}{3} b^3 + \frac{b^5}{16h^2} + \left(\frac{4h^2 - b^2}{4h} \right)^3 \log \left(\frac{2h+b}{2h-b} \right) \right].$$

11147. (Professor Zerr.)—Find the average area of a triangle formed by joining two random points in a given triangle with its vertex.

Solution.

Let CAB be the given triangle. Draw CL parallel to AB. Let R, S be the random points; through R, S draw EE′, DD′ parallel to CL. Let

$$CE = x,\quad CD = w,\quad ER = y,$$
$$DS = z,\quad EE' = y',$$
$$DD' = z',\quad DD_1 = z''.$$



Then we have

$$\text{area CRS} = \tfrac{1}{2}(wy - xz)\sin A = u,\quad \text{when } z < z'';$$
$$\text{area CRS} = \tfrac{1}{2}(xz - wy)\sin A = u_1,\quad \text{when } z > z'',$$
$$y' = \frac{cx}{b},\quad z' = \frac{cw}{b},\quad z'' = \frac{wy}{x}.$$

The required average area is

$$\Delta = \frac{\int_0^b\int_0^x\int_0^{y'}\left\{\int_0^{z''} u\,dz + \int_{z''}^{z'} u_1\,dz\right\}dx\,dw\,dy}{\int_0^b\int_0^x\int_0^{y'}\int_0^{z'} dx\,dw\,dy\,dz}$$

$$= \frac{8}{c^2b^2}\int_0^b\int_0^x\int_0^{y'}\left\{\int_0^{z''} u\,dz + \int_{z''}^{z'} u_1\,dz\right\}dx\,dw\,dy$$

$$= \frac{2\sin A}{c^2b^2}\int_0^b\int_0^x\int_0^{y'}\left\{\frac{2y^2}{x} + \frac{c^2x}{b^2} - \frac{2cy}{b}\right\}w^2\,dx\,dw\,dy$$

$$= \frac{4c\sin A}{3b^5}\int_0^b\int_0^x x^2w^2\,dx\,dw$$

$$= \frac{4c\sin A}{9b^5}\int_0^b x^5\,dx = \frac{2bc\sin A}{27} = \frac{4}{27}\text{ (area of triangle)}.$$

11148. (Professor ZERR.)—Find the average area of a triangle formed by joining three random points in the surface of a given triangle.

Solution.

Let M, N, P be the three random points; through M, N, P draw FF′, GG′, HH′ parallel to CL, GG′ cutting PM at K. Let

$$CH = u,\quad CF = v,\quad CG = w,\quad PH = y,\quad MF = x,\quad GN = z,$$
$$GK = z'',\quad FF' = x',\quad GG' = z',\quad HH' = y',\quad AC = b,\quad AB = c.$$

Then we have

$$\text{area MPN} = \tfrac{1}{2}\left[x(u-w)+y(w-v)+z(v-u)\right]\sin A = B,$$
when $z < z''$;

$$\text{area MPN} = \tfrac{1}{2}\left[x(w-u)+y(v-w)+z(u-v)\right]\sin A = B_1,$$
when $z > z''$;

$$x' = \frac{cv}{b},\quad y' = \frac{cu}{b},\quad z' = \frac{cu}{b},\quad z'' = \frac{x(u-w)+y(w-v)}{u-v}.$$

The limits of u are 0 and b', of v, 0 and u; of w, v and u; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'', and z'' and z'.

Hence the required average area is

$$\Delta = \frac{\int_0^b\int_0^u\int_v^u\int_0^{x'}\int_0^{y'}\left\{\int_0^{z''} B\,dz + \int_{z''}^{z'} B_1\,dz\right\} du\,dv\,dw\,dx\,dy}{\int_0^b\int_0^u\int_v^u\int_0^{x'}\int_0^{y'}\int_0^{z'} du\,dv\,dw\,dx\,dy\,dz}$$

$$= \frac{48}{c^3b^3}\int_0^b\int_0^u\int_v^u\int_0^{x'}\int_0^{y'}\left\{\int_0^{z''} B\,dz + \int_{z''}^{z'} B_1\,dz\right\} du\,dv\,dw\,dx\,dy$$

$$= \frac{12\sin A}{b^3c^3}\int_0^b\int_0^u\int_v^u\int_0^{x'}\int_0^{y'}\left\{\left[x(u-w)+y(w-v)\right]^2 + \left[x(u-w)+y(w-v)+\frac{cw}{b}(v-u)\right]^2\right\} du\,dv\,dw\,\frac{dx\,dy}{u-v}$$

$$= \frac{4\sin A}{b^3c^2}\int_0^b\int_0^u\int_v^u\int_0^{x'}\left\{6x^2(u-w)^2 + 6\frac{cu}{b}(u-w)(w-v) + 6\frac{cu}{b}(u-w)(v-u) + 2\frac{c^2u^2}{b^2}(w-v)^2 + 3\frac{c^2w^2}{b^2}(v-u)^2 + 3\frac{c^2uw}{b^2}(w-v)(v-u)\right\}\cdot\frac{u}{u-v}\,du\,dv\,dw\,dx$$

$$= \frac{4c\sin A}{b^7}\int_0^b\int_0^u\int_v^u\left\{2v^2(u-w)^2 + 2u^2(w-v)^2 + 3uv(u-w)(w-v) + 3w^2(v-u)^2 + 3vw(u-w)(v-u) + 3uw(w-v)(v-u)\right\}\frac{uv}{u-v}\,du\,dv\,dw$$

$$= \frac{2c\sin A}{3b^7}\int_0^b\int_0^u\left(4u^5v + 4uv^5 + 2u^3v^3 - 5u^4v^2 - 5u^2v^4\right)du\,dv$$

$$= \frac{c\sin A}{3b^7}\int_0^b u^7\,du = \frac{bc\sin A}{24} = \frac{1}{12}\,(\text{area of given triangle}).$$

[The two preceding Questions have been solved, though not so satisfactorily, by various other methods.]

11149. (Professor ZERR.)—From the cusp of a given cardioid a projectile is thrown at random with a given velocity, which is such that the

greatest range of the projectile is equal to $2a$, where $r = a(1+\cos\theta)$ is the equation to the cardioid; find the chance of its falling within the cardioid.

Solution.

Let ϕ = angle of projection, θ = direction of projection measured from OA, v = velocity of projection. Then we have

$$\mathrm{OA} = 2a, \quad \angle\,\mathrm{BOA} = \theta, \quad \text{range} = \frac{v^2}{g}\sin 2\phi = \mathrm{OB} = a(1+\cos\theta),$$

when the projectile falls in the cardioid. The range is greatest when $\phi = 45$, and then we have $\dfrac{v^2}{g} = 2a$. Hence $2\sin 2\phi = (1+\cos\theta)$, and

$$\sin\phi = \tfrac{1}{2}\left[1+\tfrac{1}{2}(1+\cos\theta)\right]^{\frac{1}{2}} \pm \tfrac{1}{2}\left[1-\tfrac{1}{2}(1+\cos\theta)\right]^{\frac{1}{2}};$$

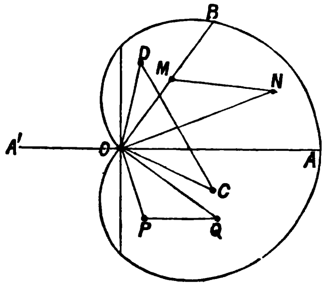
therefore the projectile will fall without the cardioid, if $\sin\phi$ is less than

$$\tfrac{1}{2}\left[1+\tfrac{1}{2}(1+\cos\theta)\right]^{\frac{1}{2}} - \tfrac{1}{2}\left[1-\tfrac{1}{2}(1+\cos\theta)\right]^{\frac{1}{2}};$$

but if $\sin\phi$ is greater than

$$\tfrac{1}{2}\left[1+\tfrac{1}{2}(1+\cos\theta)\right]^{\frac{1}{2}} + \tfrac{1}{2}\left[1-\tfrac{1}{2}(1+\cos\theta)\right]^{\frac{1}{2}},$$

the projectile will fall within the cardioid.



If all possible directions are equally probable, the chance of the projectile falling within the cardioid is

$$1 - \left[1-\tfrac{1}{2}(1+\cos\theta)\right]^{\frac{1}{2}} = 1 - \sin\tfrac{1}{2}\theta.$$

Hence the required chance is

$$p = \frac{\int_0^{2\pi}(1-\sin\frac{1}{2}\theta)\,d\theta}{\int_0^{2\pi}d\theta} = 1 - \frac{2}{\pi}.$$

The chance of its falling without the cardioid is

$$p_1 = \frac{\int_0^{2\pi}\sin\frac{1}{2}\theta\,d\theta}{\int_0^{2\pi}d\theta} = \frac{2}{\pi}.$$

11150. (Professor Zerr.)—Find the average area of the triangle formed by joining two random points, one on each side of the initial line of a cardioid, with the cusp.

Solution.

Let C, D be the random points. Let

$$OD = x, \quad OC = y, \quad \angle AOD = \theta, \quad \angle AOC = \phi.$$

Then we have $\quad$ area $ODC = \frac{1}{2}xy\sin(\theta+\phi)$.

The limits of θ are 0 and π; of ϕ, 0 and π; of x, 0 and $a(1+\cos\theta) = r$; of y, 0 and $a(1+\cos\phi) = r'$; hence the required average is

$$\Delta = \frac{\int_0^\pi\int_0^\pi\int_0^r\int_0^{r'} \frac{1}{2}xy\sin(\theta+\phi)\,d\theta\,d\phi\,x\,dx\,y\,dy}{\int_0^\pi\int_0^\pi\int_0^r\int_0^{r'} d\theta\,d\phi\,x\,dx\,y\,dy}$$

$$= \frac{16}{9\pi^2a^4}\int_0^\pi\int_0^\pi\int_0^r\int_0^{r'} \frac{1}{2}xy\sin(\theta+\phi)\,d\theta\,d\phi\,x\,dx\,y\,dy$$

$$= \frac{8}{27\pi^2a}\int_0^\pi\int_0^\pi\int_0^r \sin(\theta+\phi)(1+\cos\phi)^3\,d\theta\,d\phi\,x^2\,dx$$

$$= \frac{8a^2}{81\pi^2}\int_0^\pi\int_0^\pi \sin(\theta+\phi)(1+\cos\theta)^3(1+\cos\phi)^3\,d\theta\,d\phi$$

$$= \frac{a^2}{81\pi^2}\int_0^\pi (15\pi\sin\theta + 32\cos\theta)(1+\cos\theta)^3\,d\theta = \frac{40a^2}{27\pi}.$$

11151. (Professor Zerr.)—Find the average area of a triangle formed by joining two random points in the half of a cardioid above the initial line, to the cusp.

Solution.

Let M, N be the two random points,

$$OM = x, \quad ON = y, \quad \angle MOA = \theta, \quad \angle NOA = \phi.$$

Then the area $\quad MON = \frac{1}{2}xy\sin(\theta-\phi)$.

The limits of θ are 0 and π; of ϕ, 0 and θ; of x, 0 and $a(1+\cos\theta) = r$; of y, 0 and $a(1+\cos\phi) = r'$; hence the average area is

$$\Delta = \frac{\int_0^\pi\int_0^\theta\int_0^r\int_0^{r'} \frac{1}{2}xy\sin(\theta-\phi)\,d\theta\,d\phi\,x\,dx\,y\,dy}{\int_0^\pi\int_0^\theta\int_0^r\int_0^{r'} d\theta\,d\phi\,x\,dx\,y\,dy}$$

$$= \frac{32}{9\pi^2a^4}\int_0^\pi\int_0^\theta\int_0^r\int_0^{r'} \frac{1}{2}xy\sin(\theta-\phi)\,d\theta\,d\phi\,x\,dx\,y\,dy$$

$$= \frac{16}{27\pi^2a}\int_0^\pi\int_0^\theta\int_0^r \sin(\theta-\phi)(1+\cos\phi)^3\,d\theta\,d\phi\,x^2\,dx$$

$$\Delta = \frac{16a^2}{81\pi^2}\int_0^\pi\int_0^\theta \sin(\theta-\phi)(1+\cos\theta)^3(1+\cos\phi)^3\,d\theta\,d\phi$$
$$= \frac{2a^2}{81\pi^3}\int_0^\pi (24+15\theta\sin\theta-15\cos\theta-8\cos^2\theta-\cos^3\theta)(1+\cos\theta)^3\,dx = \frac{361a^2}{432\pi}.$$

11152. (Professor ZERR.)—Find the average area of a triangle formed by joining two random points anywhere in a cardioid to its cusp.

Solution.

Let P, Q be the two random points. Let

$$OQ = x,\quad OP = y,\quad \angle A'OQ = \theta,\quad \angle A'OP = \phi.$$

Then $\qquad$ area POQ $= \frac{1}{2}xy\sin(\theta-\phi)$.

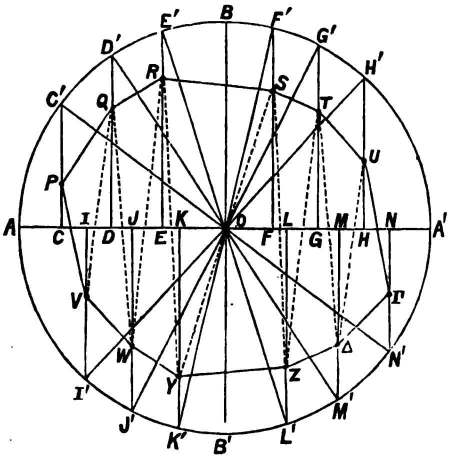
The limits of θ are 0 and 2π; of ϕ, 0 and θ; of x, 0 and $a(1-\cos\theta) = r$; of y, 0 and $a(1-\cos\phi) = r'$. The average area is

$$\Delta = \frac{\int_0^{2\pi}\int_0^\theta\int_0^r\int_0^{r'} \frac{1}{2}xy\sin(\theta-\phi)\,d\theta d\phi\,x\,dx\,y\,dy}{\int_0^{2\pi}\int_0^\theta\int_0^r\int_0^{r'} d\theta d\phi\,x\,dx\,y\,dy}$$
$$= \frac{8}{9\pi^2a^4}\int_0^{2\pi}\int_0^\theta\int_0^r\int_0^{r'} \frac{1}{2}xy\sin(\theta-\phi)\,d\theta d\phi\,x\,dx\,y\,dy$$
$$= \frac{4}{27\pi^2a}\int_0^{2\pi}\int_0^\theta\int_0^r \sin(\theta-\phi)(1-\cos\phi)^3\,d\theta d\phi\,x^2dx$$
$$= \frac{4a^2}{81\pi^2}\int_0^{2\pi}\int_0^\theta \sin(\theta-\phi)(1-\cos\theta)^3(1-\cos\phi)^3\,d\theta d\phi$$
$$= \frac{a^2}{162\pi^2}\int_0^{2\pi}(24-15\theta\sin\theta-17\cos\theta-8\cos^2\theta+\cos^3\theta)(1-\cos\theta)^3\,d\theta$$
$$= \frac{1001a^2}{864\pi}.$$

11153. (Professor ZERR.)—Twelve points are taken at random in a circle, three in each quadrant; find the average area of the dodecagon formed by joining the points.

Solution.

Let AA′, BB′ be the two perpendicular diameters dividing the surface of the circle into quadrants, P, Q, R, S, T, U, V, W, Y, Z, Δ, Γ the twelve random points taken as stated in the example. Through these points draw CC′, DD′, EE′, FF′, GG′, HH′, II′, JJ′, KK′, LL′, MM′, NN′ perpendicular to AA′.



Let $\angle C'OA = \beta$, $\angle D'OA = \gamma$, $\angle E'OA = \delta$, $\angle F'OA' = \eta$, $\angle G'OA' = \theta$, $\angle H'OA' = \lambda$, $\angle I'OA = \mu$, $\angle J'OA = \rho$, $\angle K'OA = \sigma$, $\angle L'OA' = \psi$, $\angle M'OA' = \phi$, $\angle N'OA' = \omega$,
$AO = r$, $CP = u$, $DQ = v$, $ER = w$, $FS = x$, $GT = y$, $HU = z$.
$IV = a$, $JW = b$, $KY = c$, $LZ = l$, $M\Delta = m$, $N\Gamma = n$,
$CC' = r\sin\beta = u'$, $DD' = r\sin\gamma = v'$, $EE' = r\sin\delta = w'$,
$FF' = r\sin\eta = x'$, $GG' = r\sin\theta = y'$, $HH' = r\sin\lambda = z'$,
$II' = r\sin\mu = a'$, $JJ' = r\sin\rho = b'$, $KK' = r\sin\sigma = c'$,
$LL' = r\sin\psi = l'$, $MM' = r\sin\phi = m'$, $NN' = r\sin\omega = n'$.

Area dodecagon equals the area of the ten triangles
$U\Delta\Gamma + \Delta UT + T\Delta Z + TZS + SZY + YSR + YRW + WRQ + QWV + VPQ$.

Area dodecagon

$$= \tfrac{1}{2}r\Big[u(\cos\mu-\cos\gamma)+v(\cos\beta-\cos\delta)+w(\cos\eta+\cos\gamma) \\ +x(\cos\delta+\cos\theta)+y(\cos\lambda-\cos\eta)+z(\cos\omega-\cos\theta) \\ +a(\cos\beta-\cos\rho)+b(\cos\mu-\cos\sigma)+c(\cos\psi+\cos\rho) \\ +l(\cos\phi+\cos\sigma)+m(\cos\omega-\cos\psi)+n(\cos\lambda-\cos\phi)\Big] = A.$$

An element of surface at P is $r\sin\beta\, d\beta\, du$; at Q, $r\sin\lambda\, d\lambda\, dv$; at R, $r\sin\delta\, d\delta\, dw$; at S, $r\sin\eta\, d\eta\, dx$; at T, $r\sin\theta\, d\theta\, dy$; at U, $r\sin\lambda\, d\lambda\, dz$; at V, $r\sin\mu\, d\mu\, da$; at W, $r\sin\rho\, d\rho\, db$; at Y, $r\sin\sigma\, d\sigma\, dc$; at Z, $r\sin\psi\, d\psi\, dl$; at Δ, $r\sin\phi\, d\phi\, dm$; at Γ, $r\sin\omega\, d\omega\, dn$. The limits of δ and σ are 0 and $\frac{1}{2}\pi$; of γ, 0 and δ; of β, 0 and γ; of ρ, 0 and σ; of μ, 0 and ρ; of η and ψ, 0 and $\frac{1}{2}\pi$; of θ, 0 and η; of λ, 0 and θ; of ϕ, 0 and ψ; of ω, 0 and ϕ; of u, 0 and u'; of v, 0 and v'; of w, 0 and w'; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'; of a, 0 and a'; of b, 0 and b'; of c, 0 and c'; of l, 0 and l'; of m, 0 and m'; of n, 0 and n'.

Hence the required average area is

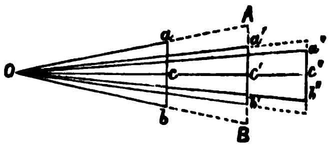
$$\Delta_1 = \left\{ \int_0^{\frac{1}{2}\pi}\int_0^{\delta}\int_0^{y}\int_0^{\frac{1}{2}\pi}\int_0^{\eta}\int_0^{\theta}\int_0^{\frac{1}{2}\pi}\int_0^{\sigma}\int_0^{\rho}\int_0^{\frac{1}{2}\pi}\int_0^{\psi}\int_0^{\phi}\int_0^{u'}\int_0^{r'}\int_0^{w'}\int_0^{x'}\int_0^{y'}\int_0^{z'}\int_0^{a'}\int_0^{b'}\int_0^{c'}\int_0^{l'}\int_0^{m'}\int_0^{n'} \right.$$
$$\times A r\sin\delta\, d\delta\, r\sin\gamma\, d\gamma\, r\sin\beta\, d\beta\, r\sin\eta\, d\eta\, r\sin\theta\, d\theta\, r\sin\lambda\, d\lambda\, r\sin\sigma\, d\sigma$$
$$\times r\sin\rho\, d\rho\, r\sin\mu\, d\mu\, r\sin\psi\, d\psi\, r\sin\phi\, d\phi\, r\sin\omega\, d\omega$$
$$\left. \times du\, dv\, dw\, dx\, dy\, dz\, da\, db\, dc\, dl\, dm\, dn \right\}$$
$$+ \left\{ \int_0^{\frac{1}{2}\pi}\int_0^{\delta}\int_0^{\gamma}\int_0^{\frac{1}{2}\pi}\int_0^{\eta}\int_0^{\theta}\int_0^{\frac{1}{2}\pi}\int_0^{\sigma}\int_0^{\rho}\int_0^{\frac{1}{2}\pi}\int_0^{\psi}\int_0^{\phi}\int_0^{u'}\int_0^{v'}\int_0^{w'}\int_0^{x'}\int_0^{y'}\int_0^{z'}\int_0^{a'}\int_0^{b'}\int_0^{c'}\int_0^{'}\int_0^{m'}\int_0^{n'} \right.$$
$$\times r\sin\delta\, d\delta\, r\sin\gamma\, d\gamma\, r\sin\beta\, d\beta\, r\sin\eta\, d\eta\, r\sin\theta\, d\theta\, r\sin\lambda\, d\lambda\, r\sin\sigma\, d\sigma$$
$$\times r\sin\rho\, d\rho\, r\sin\mu\, d\mu\, r\sin\psi\, d\psi\, r\sin\phi\, d\phi\, r\sin\omega\, d\omega$$
$$\left. \times du\, dv\, dw\, dx\, dy\, dz\, da\, db\, dc\, dl\, dm\, dn \right\}$$
$$= \frac{2^{28}.\,3^4}{\pi^{12} r^{12}}$$
$$\times \int_0^{\frac{1}{2}\pi}\int_0^{\delta}\int_0^{\gamma}\int_0^{\frac{1}{2}\pi}\int_0^{\eta}\int_0^{\theta}\int_0^{\frac{1}{2}\pi}\int_0^{\sigma}\int_0^{\rho}\int_0^{\frac{1}{2}\pi}\int_0^{\psi}\int_0^{\phi}\int_0^{u'}\int_0^{v'}\int_0^{w'}\int_0^{x'}\int_0^{y'}\int_0^{z'}\int_0^{a'}\int_0^{b'}\int_0^{c'}\int_0^{l'}\int_0^{m'}\int_0^{n'}$$
$$\times A\sin\delta\sin\gamma\sin\beta\sin\eta\sin\theta\sin\lambda\sin\sigma\sin\rho\sin\mu\sin\psi\sin\phi\sin\omega$$
$$\times d\delta\, d\gamma\, d\beta\, d\eta\, d\theta\, d\lambda\, d\sigma\, d\rho\, d\mu\, d\psi\, d\phi\, d\omega\, du\, dv\, dw\, dx\, dy\, dz\, da\, db\, dc\, dl\, dm\, dn$$
$$= \frac{2^{26}.\,3^4 r^2}{\pi^{12}} \int_0^{\frac{1}{2}\pi}\int_0^{\delta}\int_0^{\gamma}\int_0^{\frac{1}{2}\pi}\int_0^{\eta}\int_0^{\theta}\int_0^{\frac{1}{2}\pi}\int_0^{\sigma}\int_0^{\rho}\int_0^{\frac{1}{2}\pi}\int_0^{\psi}\int_0^{\phi} \left[\sin(\delta-\gamma) + \sin(\gamma-\beta)\right.$$
$$+ \sin(\delta+\eta) + \sin(\eta-\theta) + \sin(\theta-\lambda) + \sin(\sigma-\rho) + \sin(\rho-\mu)$$
$$\left. + \sin(\sigma+\psi) + \sin(\psi-\phi) + \sin(\phi-\omega) + \sin(\beta+\mu) + \sin(\lambda+\omega)\right]$$
$$\times \sin^2\delta\sin^2\gamma\sin^2\beta\sin^2\eta\sin^2\theta\sin^2\lambda\sin^2\sigma\sin^2\rho\sin^2\mu\sin^2\psi\sin^2\phi\sin^2\omega$$
$$\times d\delta\, d\gamma\, d\beta\, d\eta\, d\theta\, d\lambda\, d\sigma\, d\rho\, d\mu\, d\psi\, d\phi\, d\omega$$
$$= \frac{2^{13}.\,3^2 r^2}{\pi^6} \int_0^{\frac{1}{2}\pi}\int_0^{\delta}\int_0^{\gamma}\int_0^{\frac{1}{2}\pi}\int_0^{\eta}\int_0^{\theta} \left[\sin(\delta-\gamma) + \sin(\gamma-\beta) + \sin(\delta+\eta)\right.$$
$$\left. + \sin(\eta-\theta) + \sin(\theta-\lambda)\right] \sin^2\delta\sin^2\gamma\sin^2\beta\sin^2\eta\sin^2\theta\sin^2\lambda$$
$$\times d\delta\, d\gamma\, d\beta\, d\eta\, d\theta\, d\lambda$$
$$+ \frac{2^{19}.\,3^3 r^2}{\pi^9} \int_0^{\frac{1}{2}\pi}\int_0^{\delta}\int_0^{\gamma}\int_0^{\frac{1}{2}\pi}\int_0^{\eta}\int_0^{\theta} \left\{ \left(\frac{2\pi}{45} - \frac{409}{4725}\right)(\sin\beta + \sin\lambda)\right.$$
$$\left. + \left(\frac{\pi^2}{48} - \frac{704}{4725}\right)(\cos\beta + \cos\lambda) \right\} \sin^2\delta\sin^2\gamma\sin^2\beta\sin^2\eta\sin^2\theta\sin^2\lambda$$
$$\times d\delta\, d\gamma\, d\beta\, d\eta\, d\theta\, d\lambda$$
$$= \frac{2^{14}.\,r^2}{15\pi^6}\left\{\frac{\pi^2}{16} - \frac{704}{1575}\right\}\left\{2\pi - \frac{409}{105}\right\} + \frac{2^7.\,r^2}{\pi^3}\left\{\frac{5\pi^2}{64} - \frac{29}{45}\right\}$$
$$+ \frac{2^{12}.\,r^2}{3\pi^6}\left\{\frac{\pi^2}{32} - \frac{86}{315}\right\}\left\{\frac{7\pi}{2} - \frac{853}{105}\right\}.$$

[Prof. Zerr states that, in the foregoing part of his Appendix, he has been assisted by *The Mathematical Visitor*, an admirable *Journal*, edited by our esteemed contributor, Dr. Artemas Martin, of Washington.]

10490. (Professor ASITYA GOPHAL SARKAR.)—Give a clear and easy geometrical proof of the phenomenon that, when a ship recedes in a straight line from a spectator on the seashore, it subtends a less and less angle in the spectator's eye.

Solution.

Let O be the eye of the observer, $OCC'C''$ the straight line in which the ship moves, and $ab = a'b' = a''b''$ the different positions of the ship; then Oab, $Oa'b'$, $Oa''b''$ are isosceles triangles, and $\angle Oa'c' > OAc'$; but $\angle OAc' = Oac$, $\therefore$ $\angle Oa'c' > Oac$,



$\therefore$ $\angle Oa'c' + Ob'c' > \angle Oac + Obc$,

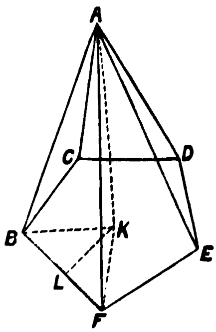
$\therefore$ $\angle a'Ob' < aOb$. Similarly, $\angle a''Ob'' < \angle a'Ob' < \angle aOb$.

It is easily proved that Oa' falls within Oa, since $Occ'c''$ bisects ab, $a'b'$, $a''b''$. From similar triangles, since $Oc' > Oc$, $AB > ab$; therefore $AB > a'b'$.

[*Otherwise*:—If ac be the ship, the angle aOc has for its tangent ac/aO; and, as ac is constant, while aO continuously increases, the angle aOc must continuously diminish.]

10981. (I. ARNOLD.)—Prove that the volume of the regular icosahedron is equal to the product obtained by multiplying the area of pentagon of icosahedron by $\frac{5}{6}$ the diameter of circumscribing sphere.

Solution.



Let a = a linear side of the icosahedron = $AB = AC = AD = BC = CD =$ &c.; R = radius of circumscribing sphere; A = area of pentagon of icosahedron. Then

$$BK = KF = a\left(\frac{5+5^{\frac{1}{2}}}{10}\right)^{\frac{1}{2}}; \quad KL = a\left(\frac{5+2\,5^{\frac{1}{2}}}{20}\right)^{\frac{1}{2}};$$

$$A = \tfrac{5}{2}a^2\left(\frac{5+2\,5^{\frac{1}{2}}}{20}\right)^{\frac{1}{2}};$$

$$R = \frac{a^2}{2AK} = \frac{a^2}{2\,(a^2 - BK^2)^{\frac{1}{2}}} = \tfrac{1}{2}a\left(\frac{5+5^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}};$$

$$D = 2R = a\left(\frac{5+5^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}; \quad \text{but}$$

$$\text{volume} = \tfrac{5}{6}a^3\left(\frac{7+3\,5^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}} = \tfrac{5}{12}a^3(3+\sqrt{5})$$

$$\left(\frac{7+3\sqrt{5}}{2}\right)^{\frac{1}{2}} = \left(\frac{14+6\sqrt{5}}{4}\right)^{\frac{1}{2}} = \tfrac{1}{2}(14+6\sqrt{5})^{\frac{1}{2}} = \tfrac{1}{2}(3+\sqrt{5});$$

hence the above expression for volume

$$= \tfrac{5}{2}a^2 \left(\frac{5+2\,5^{\frac12}}{20}\right)^{\frac12} \times \tfrac{2}{3}a \left(\frac{5+5^{\frac12}}{2}\right)^{\frac12} = \mathrm{A} \times \tfrac{2}{3}\mathrm{D}.$$

[The diameter of circumscribing sphere and radius of circle circumscribing pentagon of icosahedron are commensurable in power. If n be one-fifth of the diameter, then $5n^2 = r^3$ when r is radius of pentagon; also

$$n = \mathrm{D}/5, \quad n^2 = \mathrm{D}^2/25, \quad \text{and} \quad 5n^2 = \mathrm{D}^2/5\,; \quad \therefore\ \mathrm{D}^2 = 5r^2.]$$

10807. (Professor ORCHARD, M.A., B.Sc.)—A weight equal to that of 24 pounds, applied to a piston, forces water out of a vertical cylinder, four feet in height, through an orifice in the base, the area of the orifice being the one-hundredth of that of the base. If the mass of water initially filling the cylinder be 3 pounds, show that the time in which it will be half emptied is given by $2g^{-\frac12}(9999)^{\frac12}(18^{\frac12}-17^{\frac12})$.

Solution.

From Hydro-Mechanics we get $t = -\left(\dfrac{\mathrm{K}^2-k^2}{2gk^2}\right)^{\frac12}\displaystyle\int\frac{dx}{(x+h)^{\frac12}}$, where K = area of the surface of the liquid, k = area of the orifice, h = height of a column of water equal to the pressure applied per square inch.

By the conditions of the problem, $\mathrm{K} = 100\,k$, $h = 32$ feet;

therefore
$$t = \frac{100}{(2g)^{\frac12}}\int_2^4 \frac{dx}{(x+32)^{\frac12}} = 2g^{-\frac12}(100)\left\{18^{\frac12}-17^{\frac12}\right\}$$
$$= 2g^{-\frac12}(9999)^{\frac12}\left\{18^{\frac12}-17^{\frac12}\right\} \text{ practically.}$$

The first and second formulæ give, respectively, the values

$$t = 2g^{-\frac12}(11{\cdot}9533), \qquad t = 2g^{-\frac12}(11{\cdot}9529).$$

10773. (J. D. H. DICKSON, M.A.)—Prove that the only triangle whose sides and area are rational integers, and such that, in order, these four integers are in arithmetical progression, is that whose sides are 3 : 4 : 5.

Solution.

Let x, $x+p$, $x+2p$, $x+3p$ denote, in order, the sides and area;

then we have $\quad x^2+(x+p)^2 = (x+2p)^2, \quad \therefore\ x = 3p$ ....(1).

Also, $\quad \tfrac12 x(x+p) = x+3p$ .... (2).

This becomes, from (1), $\quad 12p^2 = 12p, \quad \therefore\ p = 1.$

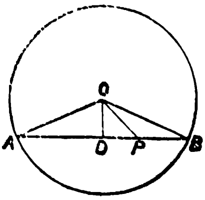
And the result is found to be as stated in problem.

2976. (Artemas Martin, LL.D.)—**A chord is drawn through a given point in the surface of a given circle; find (1) the average length of the chord, and (2) the average area of the segment cut off by the chord.**

Solution.

Let O be the centre of the given circle of radius r, P the given point, AB the chord through P,

$$AD = DB = x, \quad OP = a, \quad \angle BOD = \theta;$$



then we have $\theta = \sin^{-1}\dfrac{x}{r}$.

Area segment ADBCA is

$$r^2\left\{\sin^{-1}\frac{x}{r} - \frac{x}{r^2}(r^2-x^2)^{\frac{1}{2}}\right\},$$

$$\Delta = \text{average length of chord} = 2\int_{(r^2-a^2)^{\frac{1}{2}}}^{r} x\,dx \Big/ \int_{(r^2-a^2)^{\frac{1}{2}}}^{r} dx = r + (r^2-a^2)^{\frac{1}{2}}.$$

Δ_1 = average area of segment

$$= r^2\int_{(r^2-a^2)^{\frac{1}{2}}}^{r}\left\{\sin^{-1}\frac{x}{r} - \frac{x}{r^2}(r^2-x^2)^{\frac{1}{2}}\right\}dx \Big/ \int_{(r^2-a^2)^{\frac{1}{2}}}^{r} dx$$

$$= \left\{3r^3\pi - 2a^3 - 6ar^2 - 6r^2(r^2-a^2)^{\frac{1}{2}}\sin^{-1}\frac{(r^2-a^2)^{\frac{1}{2}}}{r}\right\} \Big/ 6\left\{r - (r^2-a^2)^{\frac{1}{2}}\right\}.$$

When $a = 0$, $\Delta = 2r$; when $a = r$, $\Delta = r$; when $a = 0$, $\Delta_1 = \frac{1}{2}\pi r^2$; when $a = r$, $\Delta_1 = \frac{1}{2}\pi r^2 - \frac{2}{3}r^2$. Also, since $2r$ = greatest, and $2(r^2-a^2)^{\frac{1}{2}}$ = least length of chord, the average length

$$= \tfrac{1}{2}\left\{2r + 2(r^2-a^2)^{\frac{1}{2}}\right\} = r + (r^2-a^2)^{\frac{1}{2}}.$$

10687. (The late Professor Seitz.)—**Two points are taken at random in the surface of a given semicircle; find the chance that the distance between them is less than the radius of the semicircle.**

Solution.

Let M be the mean distance of a point within a semicircle from a point in the opposite semicircle; M_1 the mean distance of two points taken at random anywhere within the circle; M_2 the mean distance of two points in one semicircle; then we have

$$2M_1 = M + M_2,$$

$$M = \tfrac{2}{3} - 3^{\frac{1}{2}}/2\pi \ (\textit{Math. Visitor}, \text{Vol. I., p. 32});$$

$$M_1 = 1 - 3(3)^{\frac{1}{2}}/4\pi \ (\text{Williamson's } \textit{Int. Cal.}, \text{p. 378});$$

$$\therefore\ M_2 = \tfrac{4}{3} - 3^{\frac{1}{2}}/\pi; \qquad \therefore\ M_2 = 2M = \tfrac{4}{3}M_1.$$

10789. (R. KNOWLES, B.A.)—In Question 9638, if P′, Q′, R′ be the poles of the sides of the triangle PQR, prove that the continued product of the perpendiculars on either of the asymptotes, from P, Q, R and P′, Q′, R′, respectively, are equal.

Solution.

Let $xy = a^2$ be the equation to the hyperbola, and x_1, y_1, &c., the coordinates of P, Q, R; and $2a^2/(y_1+y_2)$, $2y_1y_2/(y_1+y_2)$, &c., those of P′, Q′, R′. Then [see Solution of Quest. 9638, Vol. L., p. 136] we have

$$(y_1+y_2)(y_2+y_3)(y_1+y_3) = 8y_1y_2y_3 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1);$$

also $x_1y_1 = a^2$, $x_2y_2 = a^2$, $x_3y_3 = a^2$; $\therefore\; x_1x_2x_3y_1y_2y_3 = a^6 \;\ldots\ldots\; (2).$

The continued products of the perpendiculars from P, Q, R, and from P′, Q′, R′, on the axis of x, the one asymptote, are

$$y_1y_2y_3, \quad 8y_1^2y_2^2y_3^2/\{(y_1+y_2)(y_2+y_3)(y_1+y_3)\} = y_1y_2y_3, \text{ from (1)}.$$

The continued products from P, Q, R, and from P′, Q′, R′, on the axis of y, the other asymptote, are

$$x_1x_2x_3, \quad 8a^3/\{(y_1+y_2)(y_2+y_3)(y_1+y_3)\} = x_1x_2x_3, \text{ from (2) and (1)}.$$

Hence the theorem follows.

10536. (HUGH MACCOLL, B.A.)—A plane cuts a sphere of given radius in a random diιection. Find (1) the average volume of the smaller segment; (2) the average area of its circular base; (3) the average area of its other surface.

Solution.

Let x be the distance of the plane from the centre of the sphere, r the radius of the sphere.

Then the volume of the smaller segment is $\frac{1}{3}\pi(2r^3-3r^2x-x^3)$.

The radius of its base is $(r^2-x^2)^{\frac{1}{2}}$, and the area of its circular base is $\pi(r^2-x^2)$, and the area of its other surface is $2\pi r(r-x)$.

Hence (1) $\Delta = \left(\int_0^r \frac{1}{3}\pi(2r^3-3r^2x+x^3)\,dx\right)\Big/\left(\int_0^r dx\right) = \frac{1}{4}\pi r^3;$

(2) $\Delta_1 = \left(\int_0^r \pi(r^2-x^2)\,dx\right)\Big/\left(\int_0^r dx\right) = \frac{2}{3}\pi r^2;$

(3) $\Delta_2 = \left(\int_0^r 2\pi r\,(r-x)\,dx\right)\Big/\left(\int_0^r dx\right) = \pi r^2.$

10852. (Rev. C. H. SWIFT.)—Find in how many different ways m shillings and $(m+n)$ florins can be given away to m' boys and $(m'+n')$ girls, one coin to each, where $2m+n = 2m'+n'$.

Solution.

m' boys and $m'+n'$ girls $= 2m'+n' = 2m+n$ persons. They must form themselves into a group of m for the shillings, and $m+n$ for the florins. This can be done in $(2m+n!)/(m!\,m+n!)$ ways.

8628. (Belle Easton, B.Sc.)—In $m+n$ drawings, m balls have been drawn white, and n black, nothing else being known; show that the chance that the next two drawings shall be one white and one black is

$$\{2(m+n)(n+1)\}/\{(m+n+2)(m+n+3)\}.$$

Solution.

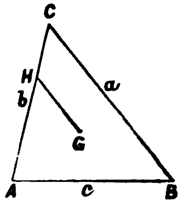
Suppose all the balls arranged along on a line AB, the white ones on the left, the black on the right of the point X where they meet.

Let AB $= 1$, AX $= x$; then the required chance is

$$h = \frac{2\int_0^1 x^{m+1}(1-x)^{n+1}\,dx}{\int_0^1 x^m (1-x)^n\,dx} = \frac{2(m+1)(n+1)}{(m+n+2)(m+n+3)}.$$

11011. (Professor Crofton, F.R.S.)—Prove that the mean distance of the vertex of a triangle from all points in the area is equal to its distance from the centroid, *measured along a parabolic path*, which leaves the vertex in the direction of one of the sides, and reaches the centroid in a direction parallel to the other, the axis of the parabola being parallel to the base.

Solution.



Let G be the centroid of the triangle CAB,

$$\text{CH} = \tfrac{1}{3}b, \quad \text{HG} = \tfrac{1}{3}a, \quad \angle\,\text{CHG} = \pi - \text{C}.$$

Then, since HG and HC are the lengths of two tangents to a parabola, and including an angle $\pi - \text{C}$, we have for the length of the arc between the points of contact (see p. 324, Ex. 1892, Wolstenholme's *Math. Problems*),

$$\tfrac{1}{3}(a+b)\left\{\frac{a^2+b^2-ab(1+\cos C)}{a^2+b^2-2ab\cos C}\right\}$$
$$+\tfrac{1}{3}\frac{a^2b^2\sin^2 C}{(a^2+b^2-2ab\cos C)^{\frac{3}{2}}}\log\frac{b}{a}\,\frac{(a^2+b^2-2ab\cos C)^{\frac{1}{2}}+b-a\cos C}{(a^2+b^2-2ab\cos C)^{\frac{1}{2}}-a+b\cos C}.$$

Substituting, in this expression, the value of cos C in terms of the sides, and $c^2 \sin^2 A$ for $a^2 \sin^2 C$, we get

$$\tfrac{1}{3}\left\{\frac{a+b}{2}+\frac{(a-b)(a^2-b^2)}{2c^2}+\frac{b^2\sin^2 A}{c}\log\frac{a+b+c}{a+b-c}\right\};$$

but $$b^2\sin^2 A = h^2 = (\text{altitude})^2.$$

Hence

$$\text{length of arc} = \tfrac{1}{3}\left\{\frac{a+b}{2}+\frac{(a-b)(a^2-b^2)}{2c^2}+\frac{h^2}{c}\log\frac{a+b+c}{a+b-c}\right\}.$$

But the same expression is the mean distance of a point within the triangle from the vertex C (see Ex. 34, p. 391 of WILLIAMSON's *Integral Calculus*).

4482. (Professor WOLSTENHOLME, Sc.D.)—Prove that (1) the equation of a cardioid may be written

$$(x+iy)^{-\frac{1}{2}}+(x-iy)^{-\frac{1}{2}}+a^{-\frac{1}{2}}=0, \quad \text{or} \quad X^{-2}+Y^{-\frac{1}{2}}+Z^{-\frac{1}{2}}=0,$$

where $i=(-1)^{\frac{1}{2}}$; and (2) that of the tricusped hypocycloid

$$(x+y\sqrt{3}+a)^{-\frac{1}{2}}+(x-y\sqrt{3}+a)^{-\frac{1}{2}}+(a-2x)^{-\frac{1}{2}}=0,$$

whence we see that either is the projection of the other.

Solution.

(1) From the equation to the cardioid

$$r = 2a(1+\cos\theta),$$

we easily get $$2ax+2a(x^2+y^2)^{\frac{1}{2}} = x^2+y^2 \quad \ldots\ldots\ldots\ldots\ldots(1);$$

dividing (1) out by a, and adding and subtracting iy to the first member,

we get $$x+iy+2(x^2+y^2)^{\frac{1}{2}}+x+iy = \frac{x^2+y^2}{a} \quad \ldots\ldots\ldots\ldots\ldots(2);$$

dividing (2) out by x^2+y^2, we get

$$\frac{1}{x+iy}+\frac{2}{(x^2+y^2)^{\frac{1}{2}}}+\frac{1}{x-iy}=\frac{1}{a} \quad \ldots\ldots\ldots\ldots\ldots(3);$$

extracting square root of (3), we get

$$\frac{1}{(x+iy)^{\frac{1}{2}}}+\frac{1}{(x-iy)^{\frac{1}{2}}}=\pm\frac{1}{a^{\frac{1}{2}}},$$

or $$(x+iy)^{-\frac{1}{2}}+(x-iy)^{-\frac{1}{2}}+a^{-\frac{1}{2}}=0.$$

(2) The equation to the tricusped hypocycloid is

$$3(x^2+y^2)^2+6a^2(x^2+y^2)-24ax^2y+8ay^3 = a^4.$$

By ordinary algebra this reduces to

$$2(a+y)(a-2y)+2(a-2y)\left\{(y+a)^2-3x^2\right\}^{\frac{1}{2}} = (y+a)^2-3x^2;$$

dividing the last equation out by $(a-2y)$, adding and subtracting $x\sqrt{3}$ to first number, we get

$$y+a+x\sqrt{3}+2\left\{(y+a)^2-3x^2\right\}^{\frac{1}{2}}+y+a-x\sqrt{3}=\frac{(y+a)^2-3x^2}{a-2y};$$

dividing the last equation out by $(y+a)^2-3x^2$,

$$\frac{1}{y+x\sqrt{3}+a}+\frac{2}{\left\{(y+a)^2-3x^2\right\}^{\frac{1}{2}}}+\frac{1}{y-x\sqrt{3}+a}=\frac{1}{a-2y};$$

extracting square root of the last,

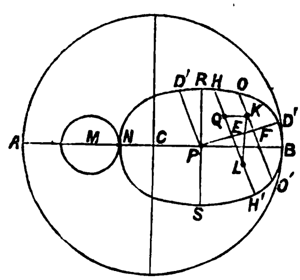
$$(y+x\sqrt{3}+a)^{-\frac{1}{2}}+(y-x\sqrt{3}+a)^{-\frac{1}{2}}+(a-2y)^{-\frac{1}{2}}=0;$$

writing x for y and y for x, the result follows.

10044. (Professor ZERR.)—From a point taken at random in the left-hand half of the horizontal diameter of a given circle, a circle is drawn at random, but so as to lie wholly in the surface of the given circle. Find (1) the average area of the triangle formed by taking three points at random in the ellipse whose major axis is that portion of the horizontal diameter between its right-hand extremity and the circumference of the random circle; and (2) the average area of the triangle formed by taking three points at random in the circle described on the same portion of the horizontal diameter, as diameter.

Solution.

Let AB be the given diameter; M the centre of the random circle; Q, L, K the three random points in the ellipse NRBS. Through Q, L draw the chord HH′; through K, draw OO′ parallel to HH′. Draw PD′ parallel to HH′, and PD conjugate to PD′.



Let $AC=a$, $AM=x$,
$MN=y$, $PR=z$, $PN=v$,
$PD'=z'$, $PD=v'$, $PE=n$,
$HL=s$, $QL=w$, $PF=m$,
$HE=EH'=u$, $OF=FO'=l$.

Then $$v=\tfrac{1}{2}(2a-x-y),\quad z'^2=\frac{z^2}{1-l^2\cos^2\theta},$$

$$u^2=\frac{z'^2}{v'^2}(v'^2-n^2),\quad \sin\phi=\frac{vz}{v'z'},\quad l^2=\frac{z'^2}{v'^2}(v'^2-m^2),$$

and the area $QLK=\tfrac{1}{2}(m-n)\,w\sin\phi$, where
$\angle D'PN=\theta$, $\angle PEH=\phi$.

An element of the ellipse at L is $\sin\phi\,dn\,ds$; at Q, $w\,d\theta\,dw$; and at K it is $2l\sin\phi\,dm$. The limits of z are 0 and v; of y, 0 and x; of x, 0 and a;

of θ, 0 and $\frac{1}{2}\pi$ and doubled; of n, $-v'$ and $+v'$; of s, 0 and $2u$; of w, 0 and s and doubled; of m, n and v' and doubled.

Hence, since the whole number of ways the three points can be taken is $\pi^3 v^3 z^3$, the required average area is

$$\Delta = \frac{8\int_0^a\int_0^x\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\int_0^s\int_n^{v'} \frac{1}{2}(m-n)\, w\sin\phi\, dx\, dy\, dz\sin\phi\, dn\, ds\, w\, d\theta\, dw \times 2l\sin\phi\, dm}{\pi^3\int_0^a\int_0^x\int_0^v v^3z^3\, dx\, dy\, dz}.$$

But $$8\int_0^a\int_0^x\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\int_0^s\int_n^{v'}(m-n)\sin^3\phi\, w^2 l\, dx\, dy\, dz\, d\theta\, dn\, ds\, dw\, dm$$

$$= \frac{2}{3}\int_0^a\int_0^x\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\int_0^s \frac{z'}{v'}\left[4(v'^2-n^2)^{\frac{3}{2}} + 6n^2(v'^2-n^2)^{\frac{1}{2}} - 3\pi v'^2 n + 6v'^2 n\sin^{-1}\frac{n}{v'}\right]\sin^3\phi\, w^2\, dx\, dy\, dz\, d\theta\, dn\, ds\, dw$$

$$= \frac{2}{9}\int_0^a\int_0^x\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\int_0^{2u}\frac{z'}{v'}\left[4(v'^2-n^2)^{\frac{3}{2}} + 6n^2(v'^2-n^2)^{\frac{1}{2}} - 3\pi v'^2 n + 6v'^2 n\sin^{-1}\frac{n}{v'}\right]\sin^3\phi\, s^3\, dx\, dy\, dz\, d\theta\, dn\, ds$$

$$= \frac{8}{9}\int_0^a\int_0^x\int_0^v\int_0^{\frac{1}{2}\pi}\int_{-v'}^{+v'}\frac{z'^5}{v'^5}\left[4(v'^2-n^2)^{\frac{3}{2}} + 6n^2(v'^2-n^2)^{\frac{1}{2}} - 3\pi v'^2 n + 6v'^2 n\sin^{-1}\frac{n}{v'}\right]\sin^3\phi\, dx\, dy\, dz\, d\theta\, dn$$

$$= \frac{35\pi v^3 z^3}{24}\int_0^a\int_0^x\int_0^v\int_0^{\frac{1}{2}\pi} z'^2\, dx\, dy\, dz\, d\theta = \frac{35\pi^2}{48}\int_0^a\int_0^x\int_0^v v^4 z^4\, dx\, dy\, dz;$$

$$\therefore\quad \Delta = \frac{35}{48\pi}\,\frac{\int_0^a\int_0^x\int_0^v v^4z^4\, dx\, dy\, dz}{\int_0^a\int_0^x\int_0^v v^3z^3\, dx\, dy\, dz} = \frac{7}{48\pi}\,\frac{\int_0^a\int_0^x (2a-x-y)^9\, dx\, dy}{\int_0^a\int_0^x (2a-x-y)^7\, dx\, dy} = \frac{651a^2}{1700\pi}.$$

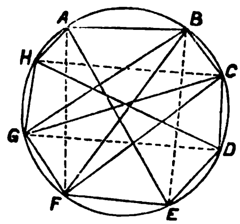
For the circle described on NB as diameter, we get, since

$$v' = v = z' = z,$$

$$\Delta_1 = \frac{35}{48\pi}\,\frac{\int_0^a\int_0^x v^8\, dx\, dy}{\int_0^a\int_0^x v^6\, dx\, dy} = \frac{35}{192\pi}\,\frac{\int_0^a\int_0^x (2a-x-y)^8\, dx\, dy}{\int_0^a\int_0^x (2a-x-y)^6\, dx\, dy} = \frac{25039a^2}{54864\pi}.$$

10949. (R. Knowles, B.A. Suggested by Question 10515.)—If an octagon inscribed in a circle has its opposite sides parallel, prove that its four diagonals will be equal.

Solution.



The quadrilaterals ABEF, GBCF, HCDG, are isosceles trapezoids, and their diagonals are equal; therefore

$$AE = BF, \quad GC = BF, \quad HD = GC;$$

therefore $AE = BF = GC = HD$.

This is true for any $2n$-gon inscribed in a circle, and having its opposite sides parallel. For since parallel chords intercept on the circumference equal arcs, the opposite sides are the parallel sides of isosceles trapezoids, and their diagonals are equal. In this way the diagonals of all the trapezoids thus formed (being the diagonals of the polygon), can be shown to be equal to each other.

11015. (The Editor.)—Prove that the lengths of the perpendiculars from the vertices A, B, C of a triangle on the line that joins the in- and circum-centres of a triangle are proportional to

$$\sin B \sin 2C - \sin 2B \sin C : \sin C \sin 2A - \sin 2C \sin A$$
$$: \sin A \sin 2B - \sin 2A \sin B.$$

Solution.

The equation to the line, in trilinear coordinates, is

$$(\cos C - \cos B)\,\alpha + (\cos A - \cos C)\,\beta + (\cos B - \cos A)\,\gamma = 0.$$

The perpendiculars on this line from the points

$$\left(\frac{\Delta}{r \sin A}, 0, 0\right), \quad \left(0, \frac{\Delta}{r \sin B}, 0\right), \quad \left(0, 0, \frac{\Delta}{r \sin C}\right),$$

are $$\frac{\Delta(\cos C - \cos B)}{r \sin A\,(A^2+B^2)^{\frac{1}{2}}}, \quad \frac{\Delta(\cos A - \cos C)}{r \sin B\,(A^2+B^2)^{\frac{1}{2}}}, \quad \frac{\Delta(\cos B - \cos A)}{r \sin C\,(A^2+B^2)^{\frac{1}{2}}};$$

where the expression $(A^2+B^2)^{\frac{1}{2}}$ has its usual significance. Hence the perpendiculars are proportional to

$$\frac{\cos C - \cos B}{\sin A} : \frac{\cos A - \cos C}{\sin B} : \frac{\cos B - \cos A}{\sin C},$$

or $$2 \sin C \sin B (\cos C - \cos B) : 2 \sin A \sin C (\cos A - \cos C)$$
$$: 2 \sin A \sin B (\cos B - \cos A);$$

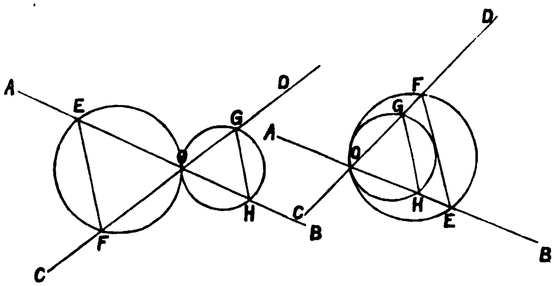
or $$\sin B \sin 2C - \sin 2B \sin C : \sin C \sin 2A - \sin 2C \sin A$$
$$: \sin A \sin 2B - \sin 2A \sin B.$$

[The altitudes of AOI, BOI, COI vary as the areas AOI, BOI, COI, or as $T.V\alpha(a\alpha + b\beta + c\gamma)$, &c., or $T.V(ba\beta - c\gamma a)$, that is to say as $(b \sin 2C - c \sin 2B)$, &c.]

10512. (J. J. BARNIVILLE.)—Given four lines, construct two similar triangles, each of which shall have two of the given lines as sides.

Solution.

Let m, n, o, p be the four given lines.
Let AB, CD be two intersecting straight lines.



Lay off OE $= m$, OH $= o$, OF $= n$, OG $= p$.
Through the points O, E, F, and O, G, H describe circles.
Then it is easily proven that the triangles OEF and OGH are similar.
Hence the problem is solved.

4746. (Professor HUDSON, M.A.)—A ray of light traverses a medium in which the density at any point is a function of (r, θ), the polar coordinates of the point; prove that, if μ be the refractive index,

$$\frac{\mu}{\rho} = \frac{\cos\psi}{r}\frac{d\mu}{d\theta} - \sin\psi\,\frac{d\mu}{dr},$$

where ρ is the radius of curvature of the path of the ray, and ψ the inclination of its tangent to the radius vector.

Solution.

By Art. 123, p. 112, fourth edition of PARKINSON's *Optics*, we have

$$\frac{\mu}{\rho} = \frac{d\mu}{dx}\cdot\frac{dy}{ds} - \frac{d\mu}{dy}\cdot\frac{dx}{ds}, \text{ when } \mu = \phi(x, y);$$

for $\mu = \phi(r, \theta)$, we have $x = r\cos\theta$, $y = r\sin\theta$;

hence we have $$\frac{dx}{dr} = \cos\theta, \quad \frac{dy}{d\theta} = r\cos\theta;$$

$$\frac{d\mu}{dx}\cdot\frac{dy}{ds} - \frac{d\mu}{dy}\cdot\frac{dx}{ds} = \frac{d\mu}{dr}\cdot\frac{dr}{dx}\cdot\frac{d\theta}{ds}\cdot\frac{dy}{d\theta} - \frac{d\mu}{d\theta}\cdot\frac{d\theta}{dy}\cdot\frac{dr}{ds}\cdot\frac{dx}{dr}$$

$$= r\frac{d\theta}{ds}\cdot\frac{d\mu}{dr} - \frac{1}{r}\cdot\frac{dr}{ds}\cdot\frac{d\mu}{d\theta};$$

therefore $$\frac{\mu}{\rho} = r\frac{d\theta}{ds}\cdot\frac{d\mu}{dr} - \frac{1}{r}\cdot\frac{dr}{ds}\cdot\frac{d\mu}{d\theta};$$

but $$r\frac{d\theta}{ds} = \sin\psi, \quad \frac{dr}{ds} = \cos\psi$$

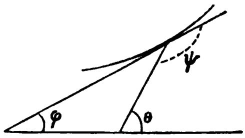
(WILLIAMSON'S *Diff. Calc.*, Art. 180);

therefore $$\frac{\mu}{\rho} = \sin\psi\frac{d\mu}{dr} - \frac{\cos\psi}{r}\cdot\frac{d\mu}{d\theta}.$$

Since the curve is convex to the pole, we must write $-\rho$ for ρ, and the result stated in the Question follows at once.

[The PROPOSER'S solution is as follows:—

Since $r^2 = x^2 + y^2$ and $\tan\theta = y/x$,
we obtain



$$\frac{dr}{dx} = \cos\theta, \quad \frac{dr}{dy} = \sin\theta,$$

$$\frac{d\theta}{dx} = -\frac{\sin\theta}{r}, \quad \frac{d\theta}{dy} = \frac{\cos\theta}{r};$$

also $$\frac{d\mu}{dx} = \frac{d\mu}{dr}\cdot\frac{dr}{dx} + \frac{d\mu}{d\theta}\cdot\frac{d\theta}{dx} = \cos\theta\frac{d\mu}{dr} - \frac{\sin\theta}{r}\cdot\frac{d\mu}{d\theta},$$

$$\frac{d\mu}{dy} = \frac{d\mu}{dr}\cdot\frac{dr}{dy} + \frac{d\mu}{d\theta}\cdot\frac{d\theta}{dy} = \sin\theta\frac{d\mu}{dr} + \frac{\cos\theta}{r}\cdot\frac{d\mu}{d\theta}.$$

Now $$\frac{dy}{ds} = \sin\phi, \quad \frac{dx}{ds} = \cos\phi;$$

therefore $$\frac{d\mu}{dx}\cdot\frac{dy}{ds} - \frac{d\mu}{dy}\cdot\frac{dx}{ds} = \sin(\phi-\theta)\frac{d\mu}{dr} - \frac{\cos(\phi-\theta)}{r}\cdot\frac{d\mu}{d\theta}.$$

But $$\theta - \phi = 180 - \psi;$$

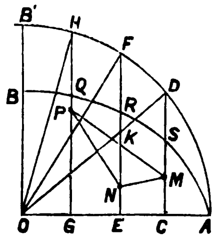
therefore $$\sin(\phi-\theta) = -\sin\psi, \quad \cos(\phi-\theta) = -\cos\psi,$$

whence we obtain the result stated in the Question.]

11010. (Professor MATZ, M.A.)—Three points are taken at random in the surface of a given elliptic (1) quadrant, (2) semi-ellipse, (3) whole ellipse; show that if a, b are the semi-axes of the ellipse, the mean area

of all the triangles that can be formed by joining the random points with straight lines is, in (1), $\frac{ab^5}{\pi}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right)$; in (2), $\frac{ab}{\pi}\left(\frac{35}{24}-\frac{32}{3\pi^2}\right)$; and in (3), $\frac{35ab}{48\pi}$.

Solution.



Let AOB be the given elliptic quadrant, AOB′ a quadrant of the circumscribing circle, M, N, P the three points; through M, N, P draw CD, EF, GH perpendicular to AO, EF intersecting MP in K. The triangle will pass through all the possible variations by considering only those relative positions of the points in which CD lies to the right of GH, and EF between CD and GH.

Let $\quad OA = a, \quad OB = b,$

$$GP = x, \quad CM = y, \quad EN = z.$$

$$GQ = x', \quad CS = y', \quad ER = z', \quad EK = z'',$$

$$\angle GOH = \theta, \quad \angle COD = \phi, \quad \angle EOF = \psi.$$

Then we have $\quad x' = b\sin\theta, \quad y' = b\sin\phi, \quad z' = b\sin\psi,$

$$v = \frac{1}{\cos\phi-\cos\theta},$$

$$z'' = v\left[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)\right].$$

$$\text{Area MNP} = \tfrac{1}{2}a\left[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)+z(\cos\theta-\cos\phi)\right] = u,$$

when $z < z''$;

$$\text{Area MNP} = \tfrac{1}{2}a\left[x(\cos\psi-\cos\phi)+y(\cos\theta-\cos\psi)+z(\cos\phi-\cos\theta)\right] = u_1,$$

when $z > z''$.

An element of surface at M is $a\sin\phi\, d\phi\, dy$, at N it is $a\sin\psi\, d\psi\, dz$, and at P it is $a\sin\theta\, d\theta\, dx$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ; of ψ, ϕ and θ; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'', and z'' and z'.

Hence the required average area is

$$\Delta = \frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\left\{\int_0^{z''}u\,dz+\int_{z''}^{z'}u_1\,dz\right\}a\sin\theta\,d\theta\,a\sin\phi\,d\phi\,a\sin\psi\,d\psi\,dx\,dy}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\int_0^{z'}a\sin\theta\,d\theta\,a\sin\phi\,d\phi\,a\sin\psi\,d\psi\,dx\,dy\,dz}$$

$$= \frac{384}{\pi^3 b^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\left\{\int_0^{z''}u\,dz+\int_{z''}^{z'}u_1\,dz\right\}\sin\theta\sin\phi\sin\psi\,d\theta\,d\phi\,d\psi\,dx\,dy$$

$$= \frac{96a}{\pi^3 b^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\left\{\left[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)\right]^2\right.$$

$$\left.+\left[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)+b\sin\psi(\cos\theta-\cos\phi)\right]^2\right\}$$

$$\times\sin\theta\sin\phi\sin\psi\, v\, d\theta\, d\phi\, d\psi\, dx\, dy$$

$$\Delta = \frac{32a}{\pi^3 b^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\Big[6x^2\sin\phi\,(\cos\phi-\cos\psi)^2$$
$$+6bx\sin^2\phi\,(\cos\phi-\cos\psi)(\cos\psi-\cos\theta)$$
$$+6bx\sin\phi\,\sin\psi\,(\cos\phi-\cos\psi)(\cos\theta-\cos\phi)$$
$$+2b^2\sin^3\phi\,(\cos\psi-\cos\theta)^2+3b^2\sin\phi\,\sin^2\psi\,(\cos\theta-\cos\phi)^2$$
$$+3b^2\sin^2\phi\,\sin\psi\,(\cos\theta-\cos\phi)(\cos\psi-\cos\theta)\Big]$$
$$\times\sin\theta\,\sin\phi\,\sin\psi\,v\,d\theta\,d\phi\,d\psi\,dx$$

$$=\frac{32ab}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\Big[2\sin^3\theta\sin\phi(\cos\phi-\cos\psi)^2+2\sin\theta\sin^3\phi\,(\cos\psi-\cos\theta)^2$$
$$+3\sin^2\theta\sin^2\phi\,(\cos\phi-\cos\psi)(\cos\psi-\cos\theta)$$
$$+3\sin\theta\,\sin\phi\,\sin^2\psi\,(\cos\theta-\cos\phi)^2$$
$$+3\sin^2\theta\sin\phi\sin\psi\,(\cos\phi-\cos\psi)(\cos\theta-\cos\phi)$$
$$+3\sin\theta\,\sin^2\phi\,\sin\psi\,(\cos\psi-\cos\theta)(\cos\theta-\cos\phi)\Big]$$
$$\times\sin\theta\,\sin\phi\,\sin\psi\,v\,d\theta\,d\phi\,d\psi$$

$$=\frac{16ab}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\Big[4\sin^2\theta\cos^2\phi+4\sin^2\phi\cos^2\phi+4\sin^2\theta\cos^2\theta+4\sin^2\phi\cos^2\theta$$
$$+\sin^2\theta\cos\theta\cos\phi+\sin^2\phi\cos\phi\cos\theta-6\sin\theta\cos\theta\sin\phi\cos\phi$$
$$+6\cos^3\theta\cos\phi+6\cos\theta\cos^3\phi+12+6\cos^2\theta+6\cos^2\phi$$
$$-36\cos\theta\cos\phi-12\sin\theta\sin\phi-9(\theta-\phi)\sin\theta\cos\phi$$
$$+9(\theta-\phi)\sin\phi\cos\theta\Big]\sin^2\theta\sin^2\phi\,d\theta\,d\phi$$

$$=\frac{8ab}{9\pi^3}\int_0^{\frac{1}{2}\pi}\Big(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta$$
$$-45\sin\theta\cos\theta-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta$$

$$=\frac{ab}{\pi}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right).$$

For the semi-ellipse above the major axis, the limits of θ are 0 and π, and those for the other variables the same as above. The number of ways the three points can be taken in the semi-ellipse is eight times the number of ways in a quadrant, and hence we get

$$\Delta_1=\frac{ab}{9\pi^3}\int_0^{\pi}\Big(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta$$
$$-45\sin\theta\cos\theta-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta$$
$$=\frac{ab}{\pi}\left(\frac{35}{24}-\frac{32}{3\pi^2}\right).$$

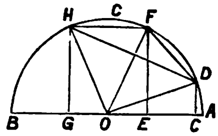
For the whole ellipse the limits of θ are 0 and 2π, and the points can

be taken eight times the number of ways in a semi-ellipse. Hence

$$\Delta_2 = \frac{ab}{72\pi^3}\int_0^{\frac{1}{2}\pi}\Big(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta$$
$$-45\sin\theta\cos\theta-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta\Big)\sin^2\theta\,d\theta$$
$$=\frac{35ab}{48\pi}.$$

10689. (Professor Nilkantha Sarkar, M.A.)—On a straight line AOB, whose mid-point is O, a semicircle ACB is drawn; show that, if P, Q, R be taken at random on the periphery AOBCA of the figure, the mean area of the triangle PQR will be $(\frac{3}{2}\pi^{-1}-12\pi^{-3})\,AO^2$.

Solution.



When all the points are on the semicircumference the area of the triangle is the average area of the triangle HFD; when all the points are on the diameter, the area is zero; hence the average area required is $\frac{1}{2}$ area HFD.

Draw GH, EF, CD perpendicular to AB.

Let $OA=r$, $\angle AOH=\theta$, $\angle AOD=\phi$, $\angle AOF=\psi$,

$$\text{area HFD}=\tfrac{1}{2}r^2\big[\sin\theta(\cos\psi-\cos\phi)+\sin\phi(\cos\theta-\cos\psi)$$
$$+\sin\psi(\cos\phi-\cos\theta)\big]=u.$$

An element of arc at H is $r\,d\theta$; at D, $r\,d\phi$; and at F, $r\,d\psi$. The limits of θ are 0 and π; of ϕ, 0 and θ; of ψ, θ and ϕ. Hence the required average area is

$$\Delta=\frac{\int_0^\pi\int_0^\theta\int_\phi^\theta \frac{1}{2}u\,r\,d\theta\,r\,d\phi\,r\,d\psi}{\int_0^\pi\int_0^\theta\int_\phi^\theta r\,d\theta\,r\,d\phi\,r\,d\psi}$$
$$=\frac{6}{r^3\pi^3}\int_0^\pi\int_0^\theta\int_\phi^\theta \tfrac{1}{2}u\,r\,d\theta\,r\,d\phi\,r\,d\psi$$
$$=\frac{3r^2}{2\pi^3}\int_0^\pi\int_0^\theta\big[(\theta-\phi)\sin(\phi-\theta)+2-2\sin\theta\sin\phi-2\cos\theta\cos\phi\big]\,d\theta\,d\phi$$
$$=\frac{3r^2}{2\pi^3}\int_0^\pi(2\theta-3\sin\theta+\theta\cos\theta)\,d\theta$$
$$=\frac{3(\pi^2-8)}{2\pi^3}r^2=(\tfrac{3}{2}\pi^{-1}-12\pi^{-3})\,r^2.$$

4961. (R. Tucker, M.A.)—A circle (O) through the foci of a rectangular hyperbola cuts the curve in P, Q on one branch, and P′, Q′ on the other branch; and the asymptotes in K, L, K′, L′. If QQ′ cut CO in M, and PN be the ordinate of P, prove that (1) MN = radius of (O); (2) KL touches the hyperbola; (3) the product of the radii of curvature of the hyperbola at P, Q varies as the cube of the radius (r) of (O); and (4) $CP^2 + CQ^2 = 2r^2$.

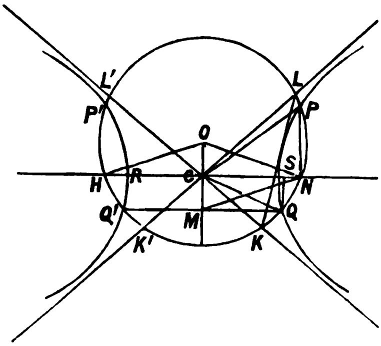
Solution.

Let $CO = b$, $CR = a$, then, if H is one of the foci,

$$CH = a\sqrt{2}, \quad OH = (b^2+2a^2)^{\frac{1}{2}} = r,$$

$$x^2 + (y-b)^2 = r^2 = 2a^2 + b^2 = \text{equation to circle},$$

$$x^2 - y^2 = a^2 = \text{equation to hyperbola}.$$



Hence the coordinates of P, Q are

$$y_1 = \tfrac{1}{2}\left[b + (2a^2+b^2)^{\frac{1}{2}}\right], \quad x_1 = \tfrac{1}{2}\left[6a^2 + 2b^2 + 2b\,(2a^2+b^2)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

$$y_2 = \tfrac{1}{2}\left[b - (2a^2+b^2)^{\frac{1}{2}}\right], \quad x_2 = \tfrac{1}{2}\left[6a^2 + 2b^2 - 2b\,(2a^2+b^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

(1) $\quad MN = (CN^2 + CM^2)^{\frac{1}{2}} = (x_1^2 - y_2^2)^{\frac{1}{2}} = (2a^2+b^2)^{\frac{1}{2}} = r.$

Also, taking QS in a positive sense, we have

$$PN + QS = \tfrac{1}{2}\left[b + (2a^2+b^2)^{\frac{1}{2}}\right] - \tfrac{1}{2}\left[b - (2a^2+b^2)^{\frac{1}{2}}\right] = (2a^2+b^2)^{\frac{1}{2}} = r.$$

(2) Since $y = x$ and $y = -x$ are the equations to LK′ and L′K, the coordinates of L are

$$y' = x' = \tfrac{1}{2}\left[b + (4a^2+b^2)^{\frac{1}{2}}\right] \quad \text{and} \quad y'' = -x'' = \tfrac{1}{2}\left[b - (4a^2+b^2)^{\frac{1}{2}}\right].$$

Hence the equation to LK is

$$by + 2a^2 = x\,(4a^2+b^2)^{\frac{1}{2}},$$

but this line meets the hyperbola at only one point $\left[\tfrac{1}{2}(4a^2+b^2)^{\frac{1}{2}},\ \tfrac{1}{2}b\right]$.

(3) For any point on the equilateral hyperbola, we have

$$\rho = (2x^2 - a^2)^{\frac{3}{2}}/a^2.$$

Hence $$\rho\rho' = \frac{(2x_1^2 - a^2)^{\frac{3}{2}}(2x_2^2 - a^2)}{a^4} = \frac{2^{\frac{3}{2}}}{a}(2a^2 + b^2)^{\frac{3}{2}} = \frac{2^{\frac{3}{2}}}{a}r^3$$

= product of radii of curvature at P, Q.

(4) $CP^2 + CQ^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 2(2a^2 + b^2) = 2r^2.$

11006. (Professor Orchard, M.A., B.Sc.)—If a right cylinder, of which the base is a loop of the curve $r = \cos 2\theta$, intercepts an area $= \pi - 2$ on the surface of a sphere, find the radius of the sphere.

Solution.

Let $x^2 + y^2 + z^2 = a^2$ be the equation to the sphere; then

$$S = \text{surface} = \iint \sec\gamma . r\,d\theta\,dr = 4a\int_0^{\frac{1}{4}\pi}\int_0^{\cos 2\theta} \frac{r}{(a^2 - r^2)^{\frac{1}{2}}}\,d\theta\,dr$$

$$= \pi a^2 - 4a\int_0^{\frac{1}{4}\pi}(a^2 - \cos^2 2\theta)^{\frac{1}{2}}\,d\theta = \pi - 2, \text{ when } a = 1.$$

Hence radius of sphere = 1.

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